Credit traps and macroprudential leverage: Online

Appendix*

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A HOUSEHOLD PREFERENCES

A commonly used form for Epstein-Zin preferences to take is:

\[ V_t = \left( c_t^{1-\rho} + \beta (\mathbb{E}_t V_{t+1}^{1-\alpha})^{\frac{1}{1-\alpha}} \right)^{\frac{1}{1-\rho}} \]  

(A.1)
in which \( \rho \) governs the degree of intertemporal substitution and \( \alpha \) the degree of risk aversion. As risk aversion goes to zero \( (\alpha \to 0) \),

\[ V_t = \left( c_t^{1-\rho} + \beta (\mathbb{E}_t V_{t+1}^{1-\alpha})^{\frac{1}{1-\alpha}} \right)^{\frac{1}{1-\rho}} \]  

(A.2)
and as \( \rho \to 1 \),

\[ V_t = c_t (\mathbb{E}_t V_{t+1})^\beta \]  

(A.3)
such that taking logs gives:

\[ U_t \equiv \log V_t = \log c_t + \beta \log \mathbb{E}_t V_{t+1} \]  

(A.4)
In the two period case, \( V_{t+1} = c_{t+1} \), so we get:

\[ U_t = \log c_t + \beta \log \mathbb{E}_t c_{t+1} \]  

(A.5)
which is the case of intra-temporal risk neutrality combined with an intertemporal elasticity of substitution of unity.
B DEPOSIT MARKET EQUILIBRIUM

B.1 Proposition 1

Proof of Proposition 1. When $R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t R_{t+1}^h$ the bank is constrained by the ex-ante pledgeability constraint (7). Given that banks are risk neutral, when $\mathbb{E}_t R_{t+1}^h > R_t^{d,h}$ they wish to raise as many deposits as they can, given they face no costs from deleveraging, in the face of a negative shock. The ex-ante pledgeability constraint then holds with equality giving

$$\lambda^h(n_t) (n_t(j) + d_t^h(j)) \mathbb{E}_t(R_{t+1}^h) = R_t^{d,h} d_t^h(j) \quad (B.1)$$

Rearranging gives the deposit demand for banks as

$$d_t^h(j) = \frac{\lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} n_t(j) > 0 \quad (B.2)$$

Turning to the terminal pay-off for banks, we have, given there are no costs from deleveraging, and using (6)

$$V_{t+1}^h(j) = \left( R_{t+1}^h - R_t^{d,h} \right) d_t^h(j) + R_{t+1}^h n_t(j) \quad (B.3)$$

Substituting in (B.2) gives

$$V_{t+1}^h(j) = \left( R_{t+1}^h - R_t^{d,h} \right) \frac{\lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} n_t(j) + R_{t+1}^h \quad (B.4)$$

This can be rearranged to $V_{t+1}^h(j) =$

$$\left[ R_{t+1}^h \left( \frac{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} + \frac{\lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)} \right) \right] n_t(j) \quad (B.5)$$
And then

\[ V_{t+1}^h(j) = \left[ \frac{R_{t+1}^h - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_t^h)} \right] R_t^{d,h} n_t(j) \]  

(B.6)

Taking time \( t \) expectations gives the required formula:

\[ \mathbb{E}_t V_{t+1}^h(j) = \frac{1 - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_t^h)} R_t^{d,h} n_t(j) > 0 \]  

(B.7)

\[ \square \]

**B.2 Proposition 2**

**Proof of Proposition 2.** Given that the conditions of Proposition 1 hold, from (B.6), the realised profits for banks are given by

\[ V_{t+1}^h(j) = \left[ \frac{R_{t+1}^h - \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h)}{R_t^{d,h} - \lambda^h(n_t) \mathbb{E}_t(R_t^h)} \right] R_t^{d,h} n_t(j) \]  

(B.8)

Thus, given \( R_t^{d,h} > \lambda^h(n_t) \mathbb{E}_t(R_t^{h}) \), banks will be solvent following a shock so long as \( R_{t+1}^h \geq \lambda^h(n_t) \mathbb{E}_t(R_{t+1}^h) \), and are thus will be solvent ex-ante with certainty when they are solvent under the lowest possible return for investment in sector \( h \), i.e., when \( R_t^{h} \geq \lambda^h(n_t) \mathbb{E}_t(R_t^{h}) \).

Given that banks are then solvent following all return realisations, and there are no costs from deleveraging, households can always withdraw deposits following a negative shock until the point at which the ex-post pledgeability constraint (10) holds, without pushing the bank into insolvency. Thus banks will not abscond with household deposits and so household deposits are indeed riskless.

\[ \square \]
B.3 Proposition 3

Proof of Proposition 3. We proceed in a number of steps.

**Step 1**: In any equilibrium we must have \( R_{t}^{d,h} > \lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h}) \).

To show this, suppose for a contradiction that it doesn’t hold: \( \lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h}) \geq R_{t}^{d,h} \). Then, given that \( \lambda^{h} (n_{t}) \in (0,1) \), and \( \mathbb{E}_{t} (R_{t+1}^{h}) > 0 \) (as \( \mathbb{E}_{t} (x_{t+1}^{h}) > 0 \)) we must have \( \mathbb{E}_{t} (R_{t+1}^{h}) > \lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h}) \geq R_{t}^{d,h} \). The *ex-ante* pledgeability constraint requires that

\[
\lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h}) (n_{t} (j) + d_{t}^{h} (j)) \geq R_{t}^{d,h} d_{t}^{h} (j)
\]

(B.9)

Under the supposition, this always holds as

\[
\lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h}) (n_{t} (j) + d_{t}^{h} (j)) \geq R_{t}^{d,h} (n_{t} (j) + d_{t}^{h} (j)) \geq R_{t}^{d,h} d_{t}^{h} (j)
\]

(B.10)

Hence, in this case the pledgeability constraint is satisfied for all \( d_{t}^{h} (j) \). Further, there is a positive spread and the bank wants to take as many deposits as possible. Given the assumption of no rationing, this cannot be an equilibrium as there is a finite amount of potential deposits available from households. Thus we have a contradiction, and so we must have \( R_{t}^{d,h} > \lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h}) \) in any equilibrium, completing the proof of Step 1.

Thus, given Step 1, banks face the pledgeability constraint. Moreover, given (20) and \( (x_{t+1}^{h})^{\alpha} > \lambda^{h} (n_{t}) \mathbb{E}_{t} ((x_{t+1}^{h})^{\alpha}) \), we also have

\[
(R_{t+1}^{h})^{\alpha} > \lambda^{h} (n_{t}) \mathbb{E}_{t} (R_{t+1}^{h})^{\alpha}
\]

(B.11)

and so deposits are riskless for households.

**Step 2**: Given \( \beta (1 - \pi) \geq \pi \), in any equilibrium we must have \( \mathbb{E}_{t} (R_{t+1}^{h}) \geq R_{t}^{d,h} \).

To show this, suppose for a contradiction that it does not hold. Then \( \mathbb{E}_{t} (R_{t+1}^{h}) < R_{t}^{d,h} \).
and banks expect to lose money on every unit of deposits taken. Optimally the bank’s deposit demand is then zero. Given this, \( E_t (V_{t+1}^h (j)) = n_t (j) E_t (R_{t+1}^h) = \pi w_t E_t (R_{t+1}^h) \) with the banks just investing their own equity. With bank deposits riskless, from (5) it then follows that household deposit demand of household \( j \) is given by

\[
d_t^h (j) = \frac{w_t}{1 + \beta} \left( \beta (1 - \pi) - \frac{\pi E_t (R_{t+1}^h)}{R_t^{d,h}} \right) > \frac{w_t}{1 + \beta} (\beta (1 - \pi) - \pi) \geq 0 \tag{B.12}
\]

where the last inequality follows from the condition and \( w_t > 0 \). Thus, household deposit supply is positive, and given the assumption of no rationing, the economy is not in equilibrium as bank deposit demand is zero. This is a contradiction, so we must have \( E_t (R_{t+1}^h) \geq R_t^{d,h} \), completing the proof of Step 2.

**Step 3:** Given \( \beta (1 - \pi) > \pi + \lambda^h (n_t) \beta \), in any equilibrium we must have \( E_t (R_{t+1}^h) > R_t^{d,h} \).

To show this, suppose for a contradiction that it does not hold. Then, given the result from Step 2, we must have \( E_t (R_{t+1}^h) = R_t^{d,h} \). Hence, with bank deposits riskless, from (B.12) we have household deposit supply given by \( d_t^h (j) = \frac{w_t}{1 + \beta} (\beta (1 - \pi) - \pi) > 0 \). Given \( E_t (R_{t+1}^h) = R_t^{d,h} \), the *ex-ante* pledgeability constraint becomes

\[
\lambda^h (n_t) (n_t (j) + d_t^h (j)) \geq d_t^h (j) \tag{B.13}
\]

Rearranging and substituting in for \( d_t^h (j) \), and using \( n_t (j) = \pi w_t \) this constraint becomes

\[
\lambda^h (n_t) \pi \geq \frac{1}{1 + \beta} (\beta (1 - \pi) - \pi) (1 - \lambda^h (n_t)) \tag{B.14}
\]

\[
\lambda^h (n_t) (\pi (1 + \beta) + (\beta (1 - \pi) - \pi)) \geq (\beta (1 - \pi) - \pi) \tag{B.15}
\]

\[
\lambda^h (n_t) \beta + \pi \geq \beta (1 - \pi) \tag{B.16}
\]

This is a contradiction, so we must have \( E_t (R_{t+1}^h) > R_t^{d,h} \), completing the proof of Step 3.
Thus, under the conditions of the proposition, in any equilibrium we must have
\[ \mathbb{E}_t \left( R^h_{t+1} \right) > R^d,h_t > \lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right) \]  
(B.17)

Thus, with bank net worth invariant to deleveraging, and (B.11) holding, the conditions of Propositions 1 and 2 hold. Thus (27) and (26) in the text hold. Equating the two for equilibrium in the deposit market gives
\[ \frac{\lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right)}{R^d,h_t - \lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right)} \pi w_t = \frac{\beta (1 - \pi) w_t - (1 - \lambda^h (n_t))}{1 + \beta} \mathbb{E}_t \left( R^h_{t+1} \right) \frac{\mathbb{E}_t \left( R^h_{t+1} \right)}{R^d,h_t - \lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right)} \pi w_t \]  
(B.18)

Cancelling common factors and rearranging gives
\[ \frac{\mathbb{E}_t \left( R^h_{t+1} \right)}{R^d,h_t - \lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right)} \left( \lambda^h (n_t) \pi + \pi \frac{1 - \lambda^h (n_t)}{1 + \beta} \right) = \frac{\beta (1 - \pi)}{1 + \beta (1 - \pi)} \]  
(B.19)

\[ \frac{\mathbb{E}_t \left( R^h_{t+1} \right)}{R^d,h_t - \lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right)} = \frac{\beta (1 - \pi)}{\pi (\beta \lambda^h (n_t) + 1)} \]  
(B.20)

Substituting this into (27) and tidying gives
\[ d^h_t = \frac{\beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \pi) w_t \]  
(B.21)

Thus, the total amount invested by banks is given by
\[ s^h_t (j) = \left( \frac{\beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \pi) + \pi \right) w_t = \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} w_t \]  
(B.22)

giving the result in the text when aggregating across banks.

Given \( \mathbb{E}_t \left( R^h_{t+1} \right) > R^d,h_t > \lambda^h (n_t) \mathbb{E}_t \left( R^h_{t+1} \right) \), from the proof of Proposition 1, in equilibrium the *ex-ante* pledgeability constraint holds with equality, so
\[ R^d,h_t = \lambda^h (n_t) \left( \frac{n_t (j)}{d^h_t (j)} + 1 \right) \mathbb{E}_t \left( R^h_{t+1} \right) \]  
(B.23)
Now, from (B.21)

\[\frac{n_t(j)}{d_t^n(j)} + 1 = \frac{(1 + \beta \lambda^h(n_t)) \pi}{\beta \lambda^h(n_t) (1 - \pi)} + 1 = \frac{\pi + \beta \lambda^h(n_t)}{\beta \lambda^h(n_t) (1 - \pi)} \] (B.24)

Thus,

\[R_{t,d,h} = \mathbb{E}_t (R_{t+1}^h) \left(\frac{\pi + \beta \lambda^h(n_t)}{\beta (1 - \pi)}\right) \] (B.25)

Given this, \(\mathbb{E}_t (R_{t+1}^h) > R_{t}^{d,h} > \lambda^h(n_t) \mathbb{E}_t (R_{t+1}^h)\) indeed holds and we have a valid equilibrium. To see this, note that, given \(\beta (1 - \pi) > \pi + \lambda^h(n_t) \beta\), from (B.25) we indeed have \(R_{t}^{d,h} < \mathbb{E}_t (R_{t+1}^h)\). Further, given \(1 + \beta \lambda^h(n_t) > 0, \pi + \beta \lambda^h(n_t) > \lambda^h(n_t) \beta (1 - \pi)\) and so we indeed have

\[R_{t}^{d,h} = \mathbb{E}_t (R_{t+1}^h) \left(\frac{\pi + \beta \lambda^h(n_t)}{\beta (1 - \pi)}\right) > \lambda^h(n_t) \mathbb{E}_t (R_{t+1}^h) \] (B.26)

Now, turning to the spread, from (B.25) we have

\[\mathbb{E}_t (R_{t+1}^h) - R_{t}^{d,h} = \mathbb{E}_t (R_{t+1}^h) \left(\frac{\beta (1 - \pi) - (\pi + \beta \lambda^h(n_t))}{\beta (1 - \pi)}\right) \] (B.27)

It remains to derive the equilibrium deposit rate. From (20), (B.22), and using \(w_t = (1 - \alpha) k^a_t\) the equilibrium return on bank’s investment, absent deleveraging, is given by

\[R_{t+1}^h = \frac{\alpha \left(\frac{x_{t+1}^h}{(1 + \beta \lambda^h(n_t)) (1 - \alpha) k^a_t} \right)^{1 - \alpha}}{\pi + \beta \lambda^h(n_t) (1 - \alpha) k^a_t} \] (B.28)

Note that as total bank returns \(R_{t+1}^h\), which include the return on capital used in the production of final output goods and liquidated capital goods, are invariant to deleveraging, (B.28) holds both when there is bank deleveraging and when there is not. Thus, from (B.25)
and taking time $t$ expectations of (B.28) we have

$$R^{d,h}_t = \frac{\alpha \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right)}{(\pi + \beta \lambda^h(n_t)) (1 - \alpha) k_t^\alpha} \left( \frac{\pi + \beta \lambda^h(n_t)}{\beta(1 - \pi)} \right)^{1-\alpha} (\pi + \beta \lambda^h(n_t))^{\alpha}$$

(B.29)

Rearranging gives

$$R^{d,h}_t = \frac{\alpha (1 + \beta \lambda^h(n_t))^{1-\alpha} (\pi + \beta \lambda^h(n_t))^{\alpha} \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right)}{\beta(1 - \pi) ((1 - \alpha) k_t^\alpha)^{1-\alpha}}$$

(B.30)

This completes the proof of the proposition.

\[\square\]

C LIQUIDATION TECHNOLOGY

In this section of the Appendix we derive the function form of liquidation technology that ensures that bank net worth is invariant to deleveraging.

**Proposition C.1.** Let $x_{t+1}^h (d_t^h + n_t)$ be the aggregate amount of capital generated by the banks’ investments with capital producers. Let $k_{t+1}^{liq}$ be the aggregate amount of liquidation of capital by banks. Let the liquidation technology be given by $y_{t+1}^{liq} = \mathbb{L} \left( k_{t+1}^{liq}, x_{t+1}^h (d_t^h + n_t) \right)$, so that the amount of output goods produced when banks liquidate $k_{t+1}^{liq}$ units of capital is dependent upon both the initial amount of aggregate capital and the amount of capital liquidated.

Suppose the liquidation technology has the following functional form:

$$\mathbb{L} \left( k_{t+1}^{liq}, x (d_t^h + n_t) \right) := \alpha (x_{t+1}^h (d_t^h + n_t))^{\alpha} - \alpha (x_{t+1}^h (d_t^h + n_t) - k_{t+1}^{liq})^{\alpha}$$

(C.1)

This functional form satisfies $\mathbb{L} \left( 0, x (d_t^h + n_t) \right) = 0$, and is increasing in $k_{t+1}^{liq}$.

Moreover, bank returns, $R_{t+1}^h$, and bank net worth, $V_{t+1}^h (j)$, are independent of the degree
of deleveraging.

Proof. It’s clear that $L(0; x^h_{t+1} (d^h_t + n_t)) = 0$, and so the liquidation technology generates no output goods when no capital is liquidated. Moreover,

$$\frac{\partial L(\ldots)}{\partial k^\text{liq}_{t+1}} = \alpha_2 \left( x^h_{t+1} (d^h_t + n_t) - k^\text{liq}_{t+1} \right)^{\alpha - 1} > 0 \quad \text{(C.2)}$$

so more output goods are produced as more capital goods are liquidated.

Turning to the second part of the proposition, in aggregate, the value of old banks at time $t + 1$ is given by

$$V^h_{t+1} = R^h_{t+1} (d^h_t + n_t) - R^d_{t+1} d^h_t \quad \text{(C.3)}$$

The banks’ investment with the capital producers generates $x^h_{t+1} (d^h_t + n_t)$ units of capital goods. Portion $k^\text{liq}_{t+1}$ is used in the production of final output goods, with the remainder, $k^\text{liq}_{t+1} := x^h_{t+1} (d^h_t + n_t) - k^\text{liq}_{t+1}$ liquidated to produce output goods. The output returned to banks from production of final output goods is given by $\alpha k^\text{liq}_{t+1}$. Thus, the value of old banks at $t + 1$, given aggregate liquidation of $k^\text{liq}_{t+1}$ units of capital, with each bank behaving symmetrically, is given by

$$V^h_{t+1} = \alpha \left( x^h_{t+1} (d^h_t + n_t) - k^\text{liq}_{t+1} \right)^{\alpha} + \mathbb{L} \left( k^\text{liq}_{t+1}, x^h_{t+1} (d^h_t + n_t) \right) - R^d_{t+1} d^h_t \quad \text{(C.4)}$$

Using the functional form in the proposition, the value of old banks is then

$$V^h_{t+1} = \alpha \left( x^h_{t+1} (d^h_t + n_t) - k^\text{liq}_{t+1} \right)^{\alpha} + \alpha \left( x^h_{t+1} (d^h_t + n_t) \right)^{\alpha} - \alpha \left( x^h_{t+1} (d^h_t + n_t) - k^\text{liq}_{t+1} \right)^{\alpha} - R^d_{t+1} d^h_t \quad \text{(C.5)}$$

$$V^h_{t+1} = \alpha \left( x^h_{t+1} (d^h_t + n_t) \right)^{\alpha} - R^d_{t+1} d^h_t \quad \text{(C.6)}$$
Thus, the aggregate value of banks, and their return on investment, is independent of the degree of deleveraging. Thus, by symmetry across banks, the value of each individual bank $V^h_{t+1}(j)$ is also independent of the degree of deleveraging. Thus so too are the returns on bank’s investments, $R^h_{t+1}$.

\[ \square \]

### D LAW OF MOTION FOR CAPITAL USED IN PRODUCTION OF FINAL GOODS

#### D.1 Proposition 4

Proof of Proposition 4. From (B.22) the total amount of capital generated by the banks capital investments is given by

\[
x^{h}_{t+1} s^h_t = x^{h}_{t+1} \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} w_t = x^{h}_{t+1} \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \alpha) k^\alpha_t
\]  

(D.1)

If $(x^{h}_{t+1})^\alpha \geq E_t ( (x^{h}_{t+1})^\alpha )$ then $R^h_{t+1} \geq E_t ( R^h_{t+1} )$ and the pledgeability constraint holds for each bank *ex-post*, and so there is no deleveraging and all the generated capital is used in the production of final output goods, giving $k_{t+1} = x^{h}_{t+1} (s^h_t)$.

Suppose instead, the capital producing technology is less productive than expected: $(x^{h}_{t+1})^\alpha < E_t ( (x^{h}_{t+1})^\alpha )$. Then the bank needs to reduce its leverage until the point at which the ex-post constraint holds. From (24), the required condition is

\[
\lambda^h (n_t) \alpha \left( x^{h}_{t+1} s^h_t - k^{\text{liq}}_{t+1} \right)^\alpha + \lambda^h \left( k^{\text{liq}}_{t+1}, x^{h}_{t+1} \left( d^h_t + n_t \right) \right) = R^{d,h}_{t} d^h_t
\]  

(D.2)

Now, given the *ex-ante* borrowing constraint holds, we have, on the basis of expected
returns
\[ \lambda^h (n_t) \alpha \mathbb{E}_t (\left( x^h_{t+1} \right)^\alpha) (s^h_t)^\alpha = R^d,h_t d^h_t \] (D.3)

Thus, substituting this in, alongside the functional form for \( L(.,.) \) from Proposition C.1

gives

\[ \lambda^h (n_t) \alpha \left( \left( x^h_{t+1} s^h_t - k^\text{liq}_{t+1} \right)^\alpha + \alpha \left( x^h_{t+1} s^h_t \right)^\alpha - \alpha \left( x^h_{t+1} s^h_t - k^\text{liq}_{t+1} \right)^\alpha \right) \] (D.4)

Gathering terms gives

\[ \left( x^h_{t+1} s^h_t - k^\text{liq}_{t+1} \right)^\alpha (1 - \lambda^h (n_t)) = \left( x^h_{t+1} s^h_t \right)^\alpha - \lambda^h (n_t) \mathbb{E}_t \left( \left( x^h_{t+1} \right)^\alpha \right) (s^h_t)^\alpha \] (D.5)

Tidying then gives

\[ \left( x^h_{t+1} s^h_t - k^\text{liq}_{t+1} \right)^\alpha = \frac{\left( x^h_{t+1} s^h_t \right)^\alpha - \lambda^h (n_t) \mathbb{E}_t \left( \left( x^h_{t+1} \right)^\alpha \right) (s^h_t)^\alpha}{(1 - \lambda^h (n_t))} \] (D.6)

Now, the capital used in the production of final output goods, \( k_{t+1} := x^h_{t+1} s^h_t - k^\text{liq}_{t+1} \).

Using (B.22) gives the aggregate capital used in the production of final output goods when \( \left( x^h_{t+1} \right)^\alpha < \mathbb{E}_t \left( \left( x^h_{t+1} \right)^\alpha \right) \) as

\[ k_{t+1} = \left( \frac{\left( x^h_{t+1} \right)^\alpha - \lambda^h (n_t) \mathbb{E}_t \left( \left( x^h_{t+1} \right)^\alpha \right)}{(1 - \lambda^h (n_t))} \right)^{\frac{\pi}{1 + \beta \lambda^h (n_t) (1 - \alpha) k^\alpha_t}} \] (D.7)
E CHOICE OF INVESTMENT BETWEEN SECTORS

E.1 Proposition 5

Proof of Proposition 5. As the conditions for Proposition 3 hold for both sectors, there would be a positive spread in each sector if it were invested in alone. Moreover, (31), (32), (34), and (B.28) all hold.

We first calculate the terminal value of bank $j$ when sector $h$ is invested in. It is assumed sector $h$ offers the highest return to deposits, so the bank raises deposits when investing in sector $h$. Its expected terminal value is given by (recalling that it’s unaffected by deleveraging)

$$V_{t+1}^h(j) = R_t^h(d_t^h(j) + n_t(j)) - R_t^{d,h}d_t^h(j) \quad (E.1)$$

From, (32) and (B.28), and using $w_t = (1 - \alpha) k_t^a$, we have

$$R_t^h(d_t^h(j) + n_t(j)) = \frac{\alpha (x_{t+1}^h)^{\alpha}}{(1 + \beta \lambda^h(n_t))^\alpha} \frac{(1 - \alpha) k_t^a}{k_t^a} \frac{(\pi + \beta \lambda^h(n_t)) (1 - \alpha) k_t^a}{(1 + \beta \lambda^h(n_t))} \quad (E.2)$$

$$= \alpha (x_{t+1})^{\alpha} \frac{(\pi + \beta \lambda^h(n_t)) (1 - \alpha) k_t^a}{1 + \beta \lambda^h(n_t)} \quad (E.3)$$

Further, from (31) and (34), and using $w_t = (1 - \alpha) k_t^a$, we have

$$R_t^{d,h} d_t^h(j) = \frac{\alpha (1 + \beta \lambda^h(n_t))^{1-\alpha}}{(1 - \alpha) k_t^a} \frac{(\pi + \beta \lambda^h(n_t))^{\alpha} \mathbb{E}_t ((x_{t+1}^h)^\alpha)}{(1 + \beta \lambda^h(n_t))} \lambda^h(n_t) (1 - \alpha) k_t^a \quad (E.4)$$

$$= \alpha \mathbb{E}_t ((x_{t+1}^h)^\alpha) \lambda^h(n_t) \frac{(\pi + \beta \lambda^h(n_t))}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^a \quad (E.5)$$

Thus, we have that

$$V_{t+1}^h(j) = \alpha \left( \frac{\pi + \beta \lambda^h(n_t)}{1 + \beta \lambda^h(n_t)} (1 - \alpha) k_t^a \right) \left[ (x_{t+1}^h)^\alpha - \mathbb{E}_t ((x_{t+1}^h)^\alpha) \lambda^h(n_t) \right] \quad (E.6)$$
And hence, the expected terminal value of bank \( j \) is given by

\[
\mathbb{E}_t V_{t+1}^h(j) = \alpha \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right) \left[ 1 - \lambda^h (n_t) \right] \left( \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha
\]  

(E.7)

Now consider the expected terminal value of bank \( j \) if it switched to investment in sector \( g \neq h \), taking no deposits (as it’s not the sector that offers the highest deposit rate to households), with every other bank investing in sector \( h \). The terminal value of this bank is given by

\[
V_{t+1}^g(j) = R_{t+1}^g n_t (j)
\]  

(E.8)

Now, \( n_t (j) = \pi (1 - \alpha) k_t^\alpha \), and the total capital goods bank \( j \) will have at the start of \( t + 1 \) will be \( x_{t+1}^g \pi (1 - \alpha) k_t^\alpha \). Note that the bank faces no pressure to de-lever as they take no deposits, so the capital generated will be used to produce output goods. When the aggregate capital used in production is given by \( k_{t+1} \) the marginal product of capital is given by \( \alpha k_{t+1}^{\alpha-1} \) thus the expected total amount of output goods returned to bank \( j \) is equal to

\[
V_{t+1}^g(j) = \mathbb{E}_t \left( \frac{\alpha x_{t+1}^g \pi (1 - \alpha) k_t^\alpha}{k_{t+1}^{1-\alpha}} \right)
\]  

(E.9)

Crucially, as the deviating bank is infinitesimal, the capital produced next period is unaltered: it is the level of investment in capital from banks investing in sector \( h \) that determines returns next period.

We consider two cases:

(i) \( R_{t}^{d,a} \geq R_{t}^{d,b} \). Then it is an equilibrium for bank \( j \) not to deviate to sector \( b \) iff

\[
\mathbb{E}_t V_{t+1}^a(j) \geq \mathbb{E}_t V_{t+1}^b(j) \quad \text{iff (noting the return on sector } b \text{ is non-stochastic)}
\]

\[
\alpha \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) \left[ 1 - \lambda^a (n_t) \right] \left( \frac{\pi + \beta \lambda^a (n_t)}{1 + \beta \lambda^a (n_t)} (1 - \alpha) k_t^\alpha \right)^\alpha \geq \alpha x^b \pi (1 - \alpha) k_t^\alpha \mathbb{E}_t \left( k_{t+1}^{\alpha-1} \right)
\]  

(E.10)
Now from (36) and (37)

$$E_t \left( k_{t+1}^{\alpha-1} \right) = \left( \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} \right)^{\alpha-1} \left( (1 - \alpha) k_t^\alpha \right)^{\alpha-1} F$$  \hspace{1cm} (E.11)

with $F$ given as in the proposition. This expression is complicated, reflecting that deleveraging by the other banks in the face of a negative shock will affect the return on capital for the deviating bank. They are not liquidating any capital goods themselves, as they have invested in sector $b$ and thus didn’t face any shocks. Indeed, in the face of a negative shock to sector $a$ returns, the deviating bank would not wish to liquidate: as the return on capital increases with less capital, their return *increases* when there is a negative shock to the other sector.

Now $E_{t+1} V^a_t(j) \geq E_{t+1} V^b_t(j)$ iff

$$E_t \left( (x_{t+1}^a)^\alpha \right) [1 - \lambda^a (n_t)] \frac{\pi + \beta \lambda^a (n_t)}{1 + \beta \lambda^a (n_t)} \geq x^b \pi F$$  \hspace{1cm} (E.12)

Now,

$$[1 - \lambda^a (n_t)] \frac{\pi + \beta \lambda^a (n_t)}{1 + \beta \lambda^a (n_t)} \geq \pi \text{ iff }$$  \hspace{1cm} (E.13)

$$[1 - \lambda^a (n_t)](\pi + \beta \lambda^a (n_t)) \geq (1 + \beta \lambda^a (n_t)) \pi \text{ iff }$$  \hspace{1cm} (E.14)

$$\pi - \pi \lambda^a (n_t) + \beta \lambda^a (n_t) - \beta (\lambda^a (n_t))^2 \geq \pi + \pi \beta \lambda^a (n_t) \text{ iff }$$  \hspace{1cm} (E.15)

$$- \pi + \beta - \beta \lambda^a (n_t) \geq \pi \beta \text{ iff }$$  \hspace{1cm} (E.16)

$$\beta (1 - \pi) \geq \pi + \beta \lambda^a (n_t)$$  \hspace{1cm} (E.17)

which is a condition of the proposition.

Thus

$$E_t \left( (x_{t+1}^a)^\alpha \right) [1 - \lambda^a (n_t)] \frac{\pi + \beta \lambda^a (n_t)}{1 + \beta \lambda^a (n_t)} \geq E_t \left( (x_{t+1}^a)^\alpha \right) \pi \geq x^b \pi F$$  \hspace{1cm} (E.18)

given that $E_t \left( (x_{t+1}^a)^\alpha \right) \geq x^b F$ is a condition of the proposition. Hence, it’s optimal for
banks not to deviate to sector $b$.

(ii) $R^{d,a}_t < R^{d,b}_t$. Then it is an equilibrium for bank $j$ not to deviate to sector $a$ iff
\[ \mathbb{E}_t V^b_{t+1}(j) \geq \mathbb{E}_t V^a_{t+1}(j) \]
iff
\[ \alpha (x^b) \alpha \left[ 1 - \lambda^b \right] \left( \frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} (1 - \alpha) k^a_t \right) \geq \mathbb{E}_t \left( \frac{\alpha x^a_{t+1} \pi (1 - \alpha) k^a_t}{k^{1-\alpha}_{t+1}} \right) \] (E.19)

noting that investment in sector $b$ is non-stochastic. Given this, the total capital used in investment by the banks investing in sector $b$ is given by
\[ k_{t+1} = x^b_{t+1} \left( \frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} \right) (1 - \alpha) k^a_t \] (E.20)

with certainty: as the returns are non-stochastic, there will be no need for deleveraging. Thus, the condition for non-deviation can be rewritten as
\[ (x^b) \alpha \left[ 1 - \lambda^b \right] \left( \frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} (1 - \alpha) k^a_t \right) \geq \mathbb{E}_t \left( \frac{\alpha x^a_{t+1} \pi (1 - \alpha) k^a_t}{k^{1-\alpha}_{t+1}} \right) \] (E.21)

\[ \left[ 1 - \lambda^b \right] \left( x^b \frac{\pi + \beta \lambda^b}{1 + \beta \lambda^b} \right) \geq \mathbb{E}_t (x^a_{t+1}) \pi \] (E.22)

which holds given the condition in the proposition. Thus, it is an equilibrium for bank $j$ not to deviate to sector $a$.

The proof of (43) then follows immediately from (34).
F EXISTENCE OF CREDIT TRAP

F.1 Conditions for existence of investment threshold \( \tilde{n} \)

**Proposition F.1.** Suppose the conditions of Proposition 5 hold. Moreover, suppose that

\[
\frac{d\lambda^a(n_t)}{dn_t} > 0 \text{ for } n_t \geq 0 \quad \text{(F.1)}
\]

\[
\mathbb{E}_t \left( (x_{t+1}^a)^{\alpha} \right) > (x^b)^{\alpha} \quad \text{(F.2)}
\]

\[
\lambda^a(0) = \tilde{\lambda}^a \in [0, \lambda^b) \quad \text{(F.3)}
\]

\[
\lim_{n_t \to \infty} \lambda^a(n_t) = \bar{\lambda}^a \in (\tilde{\lambda}^a, 1) \quad \text{(F.4)}
\]

\[
(1 + \tilde{\lambda}^a \beta)^{1-\alpha} (\pi + \tilde{\lambda}^a \beta)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^{\alpha} \right) < (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^{\alpha} (x^b)^{\alpha} \quad \text{(F.5)}
\]

\[
(1 + \bar{\lambda}^a \beta)^{1-\alpha} (\pi + \bar{\lambda}^a \beta)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^{\alpha} \right) > (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^{\alpha} (x^b)^{\alpha} \quad \text{(F.6)}
\]

Then there exists level of banker net worth \( \tilde{n} \) such that banks invest in \( a \) iff \( n_t > \tilde{n} \), where \( \tilde{n} \) is time-invariant so long as \( \mathbb{E}_t \left( (x_{t+1}^a)^{\alpha} \right) \) is constant over time. This is defined implicitly by

\[
(1 + \lambda^a (\tilde{n}) \beta)^{1-\alpha} (\pi + \lambda^a (\tilde{n}) \beta)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^{\alpha} \right) = (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^{\alpha} (x^b)^{\alpha} \quad \text{(F.7)}
\]

**Proof.** Under the conditions of the proposition, Proposition (5) holds and banks invest in
sector $a$ iff $R_t^{d,a} \geq R_t^{d,b}$. Let

$$g(n_t) := (1 + \lambda^a (n_t) \beta)^{1-\alpha} (\pi + \lambda^a (n_t) \beta)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) - (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^{\alpha} (x^b)^\alpha$$

(F.8)

Then, by Proposition (5), banks invest in sector $a$ iff $g(n_t) \geq 0$. By the above conditions, $g(0) < 0$ and $\lim_{n_t \to \infty} g(n_t) > 0$. Thus, for sufficiently large $n_t$, $g(n_t) > 0$. As $\lambda^a(.)$ is differentiable on $[0, \infty)$ it is continuous on the same interval and hence so too is $g(.)$. Thus, by the Intermediate Value Theorem, $\exists \tilde{n} : g(\tilde{n}) = 0$. Further, as $\frac{d\lambda^a(n_t)}{dn_t} > 0$ for all $n_t \geq 0$, $\frac{dg(n_t)}{dn_t} > 0$ for all $n_t \geq 0$. Hence, $\tilde{n}$ is unique, and $g(n_t) \geq 0$ iff $n_t > \tilde{n}$.

\[\Box\]

F.2 Proposition 6

Proof of Proposition 6. Given the conditions for Proposition 5 hold, banks invest in sector $b$ iff

$$ (1 + \lambda^a (n_b) \beta)^{1-\alpha} (\pi + \lambda^a (n_b) \beta)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) < (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^{\alpha} (x^b)^\alpha $$

(F.9)

Let $n^b$ be the steady state bank net worth when sector $b$ is exclusively invested in, as given by (44). Then given that under the condition of the proposition

$$ (1 + \lambda^a (n^b) \beta)^{1-\alpha} (\pi + \lambda^a (n^b) \beta)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) < (1 + \lambda^b \beta)^{1-\alpha} (\pi + \lambda^b \beta)^{\alpha} (x^b)^\alpha $$

(F.10)

at $n^b$, banks invest in sector $b$. Thus, given there are no shocks to investment returns in sector $b$, there is a steady state in which banks invest perpetually in sector $b$.

\[\Box\]
G RESILIENCE TO CREDIT TRAP

G.1 Derivation of productivity threshold for trap

From Section 3, when the economy features a credit trap, it will fall into the trap whenever bank equity falls below a critical value \( \tilde{n} \). We assume that the economy is originally investing in sector \( a \) and so is outside the credit trap. The economy will fall into a credit trap if there is a sufficiently large negative shock to the realised bank returns and consequent liquidation, leading to bank net worth falling below \( \tilde{n} \). In this case, using \( n_{t+1} = (1 - \alpha) \pi k^a_{t+1} \) and (36) applied to \( h = a \), we have the threshold productivity realisation for entering a trap as a function of \( \lambda^a, \tilde{x}^a_{t+1} (\lambda^a) \), given implicitly by

\[
\tilde{n} = (1 - \alpha) \pi \left( \frac{(\tilde{x}^a_{t+1} (\lambda^a))^\alpha - \lambda^a (n_t) \mathbb{E}_t \left( (\tilde{x}^a_{t+1})^\alpha \right)}{1 - \lambda^a (n_t)} \right) \left( \frac{\pi + \beta \lambda^a (n_t) (1 - \alpha) k^a_t}{1 + \beta \lambda^a (n_t) (1 - \alpha) k^a_t} \right) \tag{G.1}
\]

Rearranging for \( \tilde{x}^a_{t+1} (\lambda^a) \) gives (47) in the text.

G.2 Productivity trap threshold U-shaped in leverage

The following proposition demonstrates the shape of Figure 6 in the text.

**Proposition G.1.** Suppose the conditions of Proposition 5 hold. Further suppose that

\[
(1 - \alpha) \pi \mathbb{E}_t \left( (x^a_{t+1})^\alpha \right) \left\{ \frac{\pi + \beta}{1 + \beta} (1 - \alpha) k^a_t \right\} ^\alpha > \tilde{n} \tag{G.2}
\]

and

\[
\mathbb{E}_t \left( (x^a_{t+1})^\alpha \right) < \frac{\tilde{n}}{(1 - \alpha) \pi ((1 - \alpha) k^a_t)^\alpha} \left( \frac{\alpha \beta (1 - \pi) + \pi}{\pi^{1+\alpha}} \right) \tag{G.3}
\]

Then

\[
\exists \lambda^{min} \in (0, 1) : \frac{d\tilde{x}^a_{t+1} (\lambda^a)}{d\lambda^a} \begin{cases} < 0 & \text{for } \lambda^a \in (0, \lambda^{min}) \\ = 0 & \text{for } \lambda^a = \lambda^{min} \\ > 0 & \text{for } \lambda^a \in (\lambda^{min}, 1) \end{cases} \tag{G.4}
\]
Further, $\lambda^{\text{min}}$ is unique and $\tilde{x}_{t+1}^a (\lambda^a)$ reaches a unique minimum at $\lambda^a = \lambda^{\text{min}}$.

The first condition states that when there are no shocks $\left( \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) = (x_{t+1}^a)^\alpha \right)$ and $\lambda^a = 1$ the economy avoids the credit trap, i.e. if there are no shocks and leverage is high enough, it’s always possible to avoid the credit trap. The second condition ensures that when there is no borrowing ($\lambda^a = 0$), increasing leverage increases the resilience of the economy.

**Proof of Proposition G.1.** We first introduce some notation to simplify the exposition of the proof. Let

$$z(\lambda) := \lambda \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) + \frac{\tilde{n}(1 - \lambda)}{(1 - \alpha)\pi \left[ \frac{\pi + \lambda \beta}{1 + \lambda \beta} (1 - \alpha) k_t^a \right]^\alpha}$$

(G.5)

Then $\tilde{x}_{t+1}^a (\lambda) \equiv (z(\lambda))^{\frac{1}{\alpha}}$. Now $\frac{d^2 \tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} = \frac{1}{\alpha} (z(\lambda))^{\frac{1}{\alpha} - 1} z'(\lambda) > 0$ iff $z'(\lambda) > 0$. Further,

$$\frac{d^2 \tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} = \frac{1}{\alpha} (\frac{1}{\alpha} - 1) (z(\lambda))^{\frac{1}{\alpha} - 2} (z'(\lambda))^2 + \frac{1}{\alpha} (z(\lambda))^{\frac{1}{\alpha} - 1} z''(\lambda)$$

(G.6)

Hence, if $z''(\lambda) > 0$ then $\frac{d^2 \tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} > 0$. Given these results, in the following steps of the proof we can work with $z(\lambda)$. We introduce further notation: let

$$h(\lambda) := \frac{(1 - \lambda)}{\left[ \frac{\pi + \lambda \beta}{1 + \lambda \beta} \right]^\alpha}$$

(G.7)

Then

$$z(\lambda) = \lambda \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) + \frac{\tilde{n}h(\lambda)}{(1 - \alpha)\pi \left[ (1 - \alpha) k_t^a \right]^\alpha}$$

(G.8)

The proof now proceeds via a series of steps.

(i) $\frac{d^2 \tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} > 0$ for $\lambda$ close to 1. We show $z'(\lambda) > 0$ for $\lambda$ close to 1. We have

$$z'(\lambda) = \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) + \frac{\tilde{n}h'(\lambda)}{(1 - \alpha)\pi \left[ (1 - \alpha) k_t^a \right]^\alpha}$$

(G.9)
Now
\[ h'(\lambda) = -(1 + \lambda \beta)^\alpha (\pi + \lambda \beta)^{-\alpha} - \alpha \beta (1 - \pi)(1 - \lambda)(1 + \lambda \beta)^{\alpha - 1}(\pi + \lambda \beta)^{-\alpha - 1} \] (G.10)

Thus
\[ \lim_{\lambda \to 1} z'(\lambda) = \mathbb{E}_t\left( (x_{t+1}^a)^\alpha \right) - \frac{\tilde{n}}{(1 - \alpha)\pi [(1 - \alpha)k_t^\alpha]^{\alpha}} \left( \frac{\pi + \beta}{1 + \beta} \right)^\alpha \] (G.11)

This is positive so long as
\[ \mathbb{E}_t\left( (x_{t+1}^a)^\alpha \right) (1 - \alpha)\pi [(1 - \alpha)k_t^\alpha]^{\alpha} \left( \frac{\pi + \beta}{1 + \beta} \right)^\alpha > \tilde{n} \] (G.12)

Thus, given the first condition of the proposition, \( \lim_{\lambda \to 1} z'(\lambda) > 0 \). However, \( z'(\lambda) \) is continuous so \( \exists \lambda^* < 1 : z'(\lambda) > 0 \ \forall \lambda \in [\lambda^*, 1) \). Thus \( \frac{d^2 \tilde{x}_{t+1}^a(\lambda)}{d\lambda^2} > 0 \ \forall \lambda \in [0, 1] \). It is sufficient to show that \( z''(\lambda) > 0 \ \forall \lambda \in [0, 1] \). Now
\[ z''(\lambda) = \frac{\tilde{n}h''(\lambda)}{(1 - \alpha)\pi [(1 - \alpha)k_t^\alpha]^{\alpha}} \] (G.13)

Using the expression for \( h'(\lambda) \) from step (i) it can be shown that
\[ \frac{h''(\lambda)}{\alpha \beta} = \left( \frac{1 + \lambda \beta}{\pi + \lambda \beta} \right)^\alpha \left[ -1 \frac{1}{1 + \lambda \beta} + \frac{1}{\pi + \lambda \beta} \right] + (1 - \pi) \left( \frac{1 + \lambda \beta}{\pi + \lambda \beta} \right)^\alpha \left[ \frac{1}{(1 + \lambda \beta)(\pi + \lambda \beta)} + \frac{(1 - \lambda)\beta(1 - \alpha)}{(1 + \lambda \beta)^2(\pi + \lambda \beta)} + \frac{(1 - \lambda)\beta(1 + \alpha)}{(1 + \lambda \beta)(\pi + \lambda \beta)^2} \right] \] (G.14)

This expression is positive—for the first term note that \( 1 > \pi \). Hence \( z''(\lambda) > 0 \ \forall \lambda \in [0, 1] \).

(iii) We now use steps (i), (ii) to prove the proposition. We have that \( \frac{d^2 \tilde{x}_{t+1}^a(0)}{d\lambda^2} < 0 \) iff \( z'(0) < 0 \). With
\[ h'(0) = -\pi^{-\alpha} - \alpha \beta (1 - \pi)\pi^{-\alpha - 1} \] (G.15)
the condition becomes

\[ \mathbb{E}_t (\alpha x_{t+1}^\alpha) < \frac{\tilde{n} \pi^{-\alpha-1} (\pi + \alpha \beta (1 - \pi))}{(1 - \alpha) \pi [(1 - \alpha) k_t^\alpha]^{\alpha}} \]  \hspace{1cm} (G.16)

Thus by the second condition of the proposition we have \( \frac{dx_{t+1}(0)}{d\lambda} < 0 \).

Now, from step (i) \( \exists \lambda^* < 1 : \frac{dx_{t+1}(\lambda^*)}{d\lambda} > 0 \). We must have \( \lambda^* > 0 \), for otherwise, we’d have \( \frac{dx_{t+1}(0)}{d\lambda} > 0 \), a contradiction. As \( \frac{dx_{t+1}(\lambda)}{d\lambda} \) is continuous, by the Intermediate Value Theorem, \( \exists \lambda_{min} \in (0, 1) : \frac{dx_{t+1}(\lambda_{min})}{d\lambda} = 0 \). Further, as \( \frac{d^2 x_{t+1}(\lambda)}{d\lambda^2} > 0 \), \( \lambda_{min} \) is unique. The following then holds

\[
\frac{dx_{t+1}(\lambda)}{d\lambda} \begin{cases} < 0 & \text{for } \lambda \in (0, \lambda_{min}) \\ = 0 & \text{for } \lambda = \lambda_{min} \\ > 0 & \text{for } \lambda \in (\lambda_{min}, 1) \end{cases} \hspace{1cm} (G.17)
\]

And so \( \frac{dx_{t+1}(\lambda)}{d\lambda} \) reaches a unique minimum at \( \lambda = \lambda_{min} \). This completes the proof of the proposition.

G.3 Resilience maximising leverage ratio is countercyclical

**Proposition G.2.** Suppose the conditions of Proposition G.1 hold. Then, with \( \lambda_{min} \) defined as in Proposition G.1, we have that \( \lambda_{min} \) is countercyclical:

\[ \frac{d\lambda_{min}}{dk_t} < 0 \]  \hspace{1cm} (G.18)

**Proof of Proposition G.2.** Using the notation from the proof of Proposition G.1 we have
\[ \frac{d\bar{x}_{t+1}^a(\lambda)}{d\lambda} = 0 \text{ iff } z'(\lambda) = 0 \text{ iff } \]
\[ \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) = \frac{-\bar{\eta}h'(\lambda)}{(1-\alpha)\pi [(1-\alpha)k_t^\alpha]^\alpha} \quad (G.19) \]
\[ \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) \frac{(1-\alpha)\pi [(1-\alpha)k_t^\alpha]^\alpha}{\bar{n}} = \left( \frac{1 + \lambda^{\min} \beta}{\pi + \lambda^{\min} \beta} \right)^\alpha \left[ 1 + \frac{\alpha \beta (1-\pi)(1-\lambda^{\min})}{(1 + \lambda^{\min} \beta)(\pi + \lambda^{\min} \beta)} \right] \quad (G.20) \]

This last equation implicitly defines \( \lambda^{\min} \). The RHS is decreasing in \( \lambda^{\min} \). Increasing \( k_t \) increases the LHS, so \( \lambda^{\min} \) must decrease to maintain equality between the two sides of the equation. Thus \( \frac{d\lambda^{\min}}{dk_t} < 0 \), completing the proof of the proposition.

\[ \square \]

## H Welfare Functions

### H.1 Proposition 7

**Proof of Proposition 7.** Under the conditions of the proposition, the deposits by the young are given by (31), thus the consumption of the young household \( i \) is given by

\[ c_t^y (i) = (1-\pi) w_t - d_{t,t} = (1-\pi) w_t \left( 1 - \frac{\beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} \right) \quad (H.1) \]

Thus, using \( w_t = (1-\alpha) k_t^\alpha \), and noting that deposits are identical across households, we have

\[ c_t^y = (1-\alpha) k_t^\alpha \frac{(1-\pi)}{1 + \beta \lambda^h (n_t)} \quad (H.2) \]

Turning the the consumption of the old, for household \( i \) we have

\[ c_{t+1}^o (i) = R_t^{dh} d_t^h (i) + V_{t+1}^h (i) \quad (H.3) \]
Now, under the conditions of the proposition, \( R_{d,h}^t d_t^h (i) \) is given by (E.5) whilst \( V_{t+1}^h (j) \) is given by (E.6), thus

\[
c_{t+1}^o (i) = \alpha \left( \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \alpha) k_t^\alpha \right)^{\alpha} \\
\times \left[ (x_{t+1}^h)^\alpha - \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right) \lambda^h (n_t) + \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right) \lambda^h (n_t) \right]
\]

Noting that this is identical across households, and taking time \( t \) expectations, we have

\[
\mathbb{E}_t (c_{t+1}^o) = \alpha \left( \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} (1 - \alpha) k_t^\alpha \right)^{\alpha} \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right)
\]

Thus lifetime utility for the generation born at \( t \), \( U_t = \log (c_{t}^o) + \beta \log \left( \mathbb{E}_t (c_{t+1}^o) \right) \), is given by

\[
U_t = \log \left( (1 - \alpha) (1 - \pi) \right) + \alpha \log (k_t) - \log \left( 1 + \beta \lambda^h (n_t) \right) \\
+ \beta \left\{ \log (\alpha \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right) (1 - \alpha)^\alpha) + \alpha \log \left( \frac{\pi + \beta \lambda^h (n_t)}{1 + \beta \lambda^h (n_t)} \right) + \alpha^2 \log (k_t) \right\}
\]

Thus,

\[
U_t = (\alpha + \alpha^2 \beta) \log (k_t) + \alpha \beta \log \left( \pi + \beta \lambda^h (n_t) \right) - (1 + \alpha \beta) \log \left( 1 + \beta \lambda^h (n_t) \right) + C_t
\]

with \( C_t \) independent of policy and given by

\[
C_t := \log \left( (1 - \alpha) (1 - \pi) \right) + \beta \left\{ \log (\alpha \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right) (1 - \alpha)^\alpha) \right\}
\]
**H.2 Proposition 8**

*Proof of Proposition 8.* As discussed in the text, the planner’s objective at time $t$ is to maximise the discounted expected lifetime utility of each generation (excluding the old at time $t$, whose utility the planner can’t affect with the tools they have at their disposal):

$$\sum_{s=0}^{\infty} \beta^s \mathbb{E}_t U_{t+s}.$$ 

Using the results of Proposition 7 this is given by

$$\sum_{s=0}^{\infty} \beta^s \mathbb{E}_t \left\{ \left( \alpha + \alpha^2 \beta \right) \log (k_{t+s}^h) + \alpha \beta \log (\pi + \beta \lambda^h (n_{t+s})) - (1 + \alpha \beta) \log \left( 1 + \beta \lambda^h (n_{t+s}) \right) \right\}$$

$$+ \sum_{s=0}^{\infty} \beta^s \mathbb{E}_t \left\{ \log ((1 - \alpha) (1 - \pi)) + \beta \left\{ \log (\alpha \mathbb{E}_{t+s} ((x_{t+s+1}^h)^\alpha) (1 - \alpha)^\alpha) \right\} \right\}$$

Using $n_{t+s} = \pi (1 - \alpha) k_{t+s}^h$ and defining the second term as $\bar{C}$ gives the result.

$\square$

**I WELFARE SOLUTION WITHOUT CREDIT TRAPS**

In this section of the Appendix we prove Proposition 9. The proof is involved, so we proceed in several steps, first establishing sub-results.

**I.1 Lemma**

We first state & prove a useful lemma.

**Lemma I.1.** Suppose the conditions of Proposition 3 hold and the economy invests exclusively in sector $h$. Suppose further that the planner uses the haircut $\tau_s$ and

$$(1 - \tau_{t+j}) \lambda^h (n_{t+j}) \text{ is independent of } k_{t+j} \forall j \geq 1$$

(I.1)
Then, for \( j \geq 1 \)

\[
\mathbb{E}_t \frac{\partial \log (k_{t+j})}{\partial \tau_t} = \alpha^{j-1} \mathbb{E}_t \frac{\partial \log (k_{t+1})}{\partial \tau_t}
\]

(I.2)

Proof. As the conditions of Proposition 3 hold, we have, using (36) and (37), and

\[
\int_{x_{t+1}^h}^{x_{t+1}^h} f \left( x_{t+1}^h \right) dx_{t+1}^h = 1
\]

(I.3)

that, when the planner uses the haircut tool

\[
\mathbb{E}_t \log (k_{t+1}) = \alpha \log (k_t)
\]

(I.4)

\[
+ \log \left\{ \left( \frac{\pi + \beta (1 - \tau_t) \lambda^h (n_t)}{1 + \beta (1 - \tau_t) \lambda^h (n_t)} \right) (1 - \alpha) \right\} + \int_{x_{t+1}^h}^{x_{t+1}^h} \log \left( x_{t+1}^h \right) f \left( x_{t+1}^h \right) dx_{t+1}^h
\]

\[
\mathbb{E}_t \left( \left( x_{t+1}^h \right)^\alpha \right)^{\frac{1}{\hat{\alpha}}}
\]

\[
+ \int_{x_{t+1}^h}^{x_{t+1}^h} \frac{1}{\alpha} \log \left( \frac{\left( x_{t+1}^h \right)^\alpha - (1 - \tau_t) \lambda^h (n_t) \mathbb{E}_t \left( \left( x_{t+1}^h \right)^\alpha \right)}{1 - (1 - \tau_t) \lambda^h (n_t)} \right) f \left( x_{t+1}^h \right) dx_{t+1}^h
\]

Using this, we prove the lemma by induction on \( j \).

Base Case: \( j = 1 \). Then LHS (I.2) = \( \mathbb{E}_t \frac{\partial \log (k_{t+1})}{\partial \tau_t} \), whilst RHS (I.2) = \( \alpha^{1-1} \mathbb{E}_t \frac{\partial \log (k_{t+1})}{\partial \tau_t} = \mathbb{E}_t \frac{\partial \log (k_{t+1})}{\partial \tau_t} \). Thus, the base case holds.

Inductive Step. Suppose (I.2) holds \( \forall j = 1, \ldots, m \). Thus, in particular, as it holds for \( j = m \),

\[
\mathbb{E}_t \frac{\partial \log (k_{t+m})}{\partial \tau_t} = \alpha^{m-1} \mathbb{E}_t \frac{\partial \log (k_{t+1})}{\partial \tau_t}
\]

(I.5)
Consider $j = m + 1$. By using the above expression for $E_{t+m} log (k_{t+m+1})$, and taking time $t$ expectations of it, we have

\[ E_t \{ E_{t+m} log (k_{t+m+1}) \} = \alpha E_t log (k_{t+m}) \]  \tag{I.6}

\[ + E_t log \left\{ \left( \frac{\pi + \beta (1 - \tau_{t+m}) \lambda^h (n_{t+m})}{1 + \beta (1 - \tau_{t+m}) \lambda^h (n_{t+m})} \right) (1 - \alpha) \right\} \]

\[ + \int_{x_{t+m+1}^h}^{E} \log (x_{t+m+1}^h) f (x_{t+m+1}^h) dx_{t+m+1}^h \]

\[ + E_t \int_{x_{t+m+1}^h}^{E} \frac{1}{\alpha} log \left( \frac{(x_{t+m+1}^h)^\alpha - (1 - \tau_{t+m}) \lambda^h (n_{t+m}) E_{t+m} ((x_{t+m+1}^h)^\alpha)}{1 - (1 - \tau_{t+m}) \lambda^h (n_{t+m})} \right) f (x_{t+m+1}^h) dx_{t+m+1}^h \]

where $E := (E_{t+m} ((x_{t+m+1}^h)^\alpha))^{1/\alpha}$.

Now, by the condition of the lemma, $(1 - \tau_{t+j}) \lambda^h (n_{t+j})$ is independent of $k_{t+j} \forall j \geq 1$. Thus, in particular, it holds for $t + m$ and future periods, with $m \geq 1$. Thus, the only impact of $\tau_t$ on $E_t \{ E_{t+m} log (k_{t+m+1}) \}$ will come through its direct impact on $k_{t+m}$, with no indirect impact on $(1 - \tau_{t+m}) \lambda^h (n_{t+m})$, and no impact on the exogenous realisations of capital productivity $x_{t+m+1}^h$. It then follows, using (I.5), that

\[ \frac{\partial E_t \{ E_{t+m} log (k_{t+m+1}) \}}{\partial \tau_t} = \alpha \frac{\partial E_t log (k_{t+m})}{\partial \tau_t} = \alpha^{m-1} E_t \frac{\partial log (k_{t+1})}{\partial \tau_t} = \alpha^m E_t \frac{\partial log (k_{t+1})}{\partial \tau_t} \]  \tag{I.7}

By L.I.E., $E_t \{ E_{t+m} log (k_{t+m+1}) \} = E_t log (k_{t+m+1})$, and hence

\[ E_t \frac{\partial log (k_{t+m+1})}{\partial \tau_t} = \alpha^m E_t \frac{\partial log (k_{t+1})}{\partial \tau_t} \]  \tag{I.8}

Thus, the inductive step is complete, hence by the Principal of Mathematical Induction, the proof of the lemma is complete.

\[ \square \]
I.2 Optimal policy with finite horizon

We first establish optimal policy when there is a finite horizon, before turning in the next subsection to the case of an infinite horizon.

**Proposition I.1.** Suppose the conditions of Proposition 3 hold, but that the planner has a finite horizon, with terminal period $T$. Then, in all periods up to and including $T$, optimally effective pledgeability $(1 - \tau_{T-s}) \lambda^h (n_{T-s})$ is independent of $k_{T-s} \forall s \geq 0$. Moreover, the optimal haircut policy in period $T - s$ satisfies

\[
\frac{(1 + \alpha \beta) (1 - (\alpha \beta)^s + 1)}{1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s})} = \frac{\alpha \beta (2 - (1 + \alpha \beta) (\alpha \beta)^s)}{\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s})} \quad (I.9)
\]

\[
+ [1 - (\alpha \beta)^s] \int_{x_{T-s+1}^h}^{(x_{T-s+1}^h)\hat{\alpha}} \frac{(1 + \alpha \beta) \mathbb{E}_{T-s} \left( \left( x_{T-s+1}^h \right)^\alpha \right) f \left( x_{T-s+1}^h \right)}{\left( x_{T-s+1}^h \right)^\alpha - (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \mathbb{E}_{T-s} \left( \left( x_{T-s+1}^h \right)^\alpha \right)} dx_{T-s+1}^h
\]

\[
- [1 - (\alpha \beta)^s] \int_{x_{T-s+1}^h}^{(x_{T-s+1}^h)\hat{\alpha}} \frac{(1 + \alpha \beta) f \left( x_{T-s+1}^h \right)}{1 - (1 - \tau_{T-s}) \lambda^h (n_{T-s})} dx_{T-s+1}^h = 0
\]

**Proof.** We first prove that $(1 - \tau_{T-s}) \lambda^h (n_{T-s})$ is independent $k_{T-s} \forall s \geq 0$ by induction on $s$, the number of periods prior to the end of the planner’s horizon at time $T$.

**Base Case:** $s = 0$.

The end of the planner’s horizon is not the end of the world, but the last generation whose welfare they take into account when setting policy. In the terminal period the planner’s objective is to choose $\tau_T$ to maximise $U_T$ as given by (51). Noting that at time $T$, $k_T$ is given, and $C_T$ is independent of policy, their objective is

\[
\alpha \beta \log \left( \pi + (1 - \tau_T) \lambda^h (n_T) \beta \right) - (1 + \alpha \beta) \log \left( 1 + (1 - \tau_T) \lambda^h (n_T) \beta \right) \quad (I.10)
\]
The FOC w.r.t. $\tau_T$ is given by

$$\frac{-\alpha\beta \lambda^h(n_T) \beta}{\pi + (1 - \tau_T) \lambda^h(n_T) \beta} + \frac{(1 + \alpha\beta) \lambda^h(n_T) \beta}{1 + (1 - \tau_T) \lambda^h(n_T) \beta} = 0 \quad (I.11)$$

Note that this is of the same form as (I.9). Setting $s = 0$ in (I.9) gives (with the integral term disappearing)

$$\frac{(1 + \alpha\beta) (1 - (\alpha\beta))}{1 + (1 - \tau_T) \lambda^h(n_T) \beta} - \frac{\alpha\beta (2 - (1 + \alpha\beta))}{\pi + (1 - \tau_T) \lambda^h(n_T) \beta} = 0 \quad (I.12)$$

noting that $2 - (1 + \alpha\beta) = 1 - (\alpha\beta)$, and cancelling gives the same condition as (I.11), when the common factor of $\lambda^h(n_T) \beta$ has been cancelled.

Returning to the FOC, rearranging and cancelling the common $\lambda^h(n_T) \beta$ term gives

$$(1 + \alpha\beta) \left(\pi + (1 - \tau_T) \lambda^h(n_T) \beta\right) = \alpha\beta \left(1 + (1 - \tau_T) \lambda^h(n_T) \beta\right) \quad (I.13)$$

Which can be rearranged to give

$$(1 - \tau_T) \lambda^h(n_T) = \frac{\alpha\beta - (1 + \alpha\beta) \pi}{\beta} \quad (I.14)$$

This is a valid maximum, as the second derivative is negative at the FOC. To see this, note that the second derivative of the objective w.r.t. $\tau_T$ is given by

$$\frac{-\alpha\beta \left(\lambda^h(n_T) \beta\right)^2}{(\pi + (1 - \tau_T) \lambda^h(n_T) \beta)^2} + \frac{(1 + \alpha\beta) \left(\lambda^h(n_T) \beta\right)^2}{(1 + (1 - \tau_T) \lambda^h(n_T) \beta)^2} \quad (I.15)$$

The second derivative is then negative whenever

$$(1 + \alpha\beta) \left(\pi + (1 - \tau_T) \lambda^h(n_T) \beta\right)^2 < \alpha\beta \left(1 + (1 - \tau_T) \lambda^h(n_T) \beta\right)^2 \quad (I.16)$$

Using (I.13), this follows as $\left(\pi + (1 - \tau_T) \lambda^h(n_T) \beta\right) < \left(1 + (1 - \tau_T) \lambda^h(n_T) \beta\right)$, with
\[ \pi < 1. \]

Thus, in the base case, optimal effective pledgeability (and hence effective leverage), as given by (1.14) is independent of \( k \).

**Inductive Step**

Suppose \((1 - \tau_{T-s}) \lambda^h (n_{T-s})\) is independent of \( k_{T-s} \) \( \forall s = 0, ..., k \).

Consider \( s = k + 1 \). Then, from (53) the objective function of the policymaker at time \( T - k - 1 \), ignoring constants independent of policy, is given by

\[
\begin{align*}
\sum_{s=T-k-1}^{T} \beta^{s-(T-k-1)} & E_{T-k-1} \left( \alpha + \alpha^2 \beta \right) \log (k_s) \\
+ \sum_{s=T-k-1}^{T} \beta^{s-(T-k-1)} & E_{T-k-1} \left[ \alpha \beta \log \left( \pi + (1 - \tau_s) \lambda^h (n_s) \beta \right) \right] \\
- \sum_{s=T-k-1}^{T} \beta^{s-(T-k-1)} & E_{T-k-1} \left[ (1 + \alpha \beta) \log \left( 1 + (1 - \tau_s) \lambda^h (n_s) \beta \right) \right]
\end{align*}
\]

Noting that, by the inductive hypothesis, \((1 - \tau_{T-s}) \lambda^h (n_{T-s})\) is independent of \( k_{T-s} \) \( \forall s = 0, ..., k \), and hence unaffected by \( \tau_{T-k-1} \), and \( k_{T-k-1} \) is given at time \( T - k - 1 \) and so independent of policy, the FOC w.r.t. \( \tau_{T-k-1} \) is given by

\[
\begin{align*}
\left( \alpha + \alpha^2 \beta \right) & \sum_{s=T-k}^{T} \beta^{s-(T-k-1)} E_{T-k-1} \frac{\partial \log (k_s)}{\partial \tau_{T-k-1}} \\
+ \frac{(1 + \alpha \beta) \lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\alpha \beta \lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} = 0
\end{align*}
\]

Moreover, using the inductive hypothesis and Lemma I.1, for \( s \geq T - k \)

\[
E_{T-k-1} \frac{\partial \log (k_s)}{\partial \tau_{T-k-1}} = \alpha^{s-(T-k)} E_{T-k-1} \frac{\partial \log (k_{T-k})}{\partial \tau_{T-k-1}}
\]
Thus

\[
\sum_{s=T-k}^{T} \beta^{s-(T-k-1)} \mathbb{E}_{T-k} \frac{\partial \log(k_s)}{\partial \tau_{T-k-1}} = \beta \mathbb{E}_{T-k-1} \frac{\partial \log(k_{T-k})}{\partial \tau_{T-k-1}} \sum_{s=T-k}^{T} (\alpha \beta)^{s-(T-k)} \tag{I.20}
\]

Now

\[
\sum_{s=T-k}^{T} (\alpha \beta)^{s-(T-k)} = \sum_{s=0}^{k} (\alpha \beta)^{s} = 1 - (\alpha \beta)^{k+1} \tag{I.21}
\]

And so, we have

\[
(\alpha + \alpha^2 \beta) \sum_{s=T-k}^{T} \beta^{s-(T-k-1)} \mathbb{E}_{T-k} \frac{\partial \log(k_s)}{\partial \tau_{T-k-1}} = \tag{I.22}
\]

\[
\frac{\alpha \beta (1 + \alpha \beta)}{1 - \alpha \beta} \left[ 1 - (\alpha \beta)^{k+1} \right] \frac{\partial \mathbb{E}_{T-k} \log(k_{T-k})}{\partial \tau_{T-k-1}}
\]

Thus, the FOC can be written as

\[
\frac{(1 + \alpha \beta) \lambda^h (n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\alpha \beta \lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta}
\]

\[
+ \frac{\alpha \beta (1 + \alpha \beta)}{1 - \alpha \beta} \left[ 1 - (\alpha \beta)^{k+1} \right] \frac{\partial \mathbb{E}_{T-k} \log(k_{T-k})}{\partial \tau_{T-k-1}} = 0 \tag{I.23}
\]

From the proof of Lemma I.1, we have that the terms that \( \tau_{T-k-1} \) can influence in

\[
\mathbb{E}_{T-k} \log(k_{T-k})
\]

are given by

\[
\log \left( \pi + \beta (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \right) - \log \left( 1 + \beta (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \right) + \tag{I.24}
\]

\[
\frac{\mathbb{E}_{T-k} (x_{T-k}^h)^{\alpha}}{\pi} \int_{x_{T-k}^h} \log \left( \frac{(x_{T-k}^h)^{\alpha} - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} \left( (x_{T-k}^h)^{\alpha} \right)}{1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} \right) \frac{f(x_{T-k}^h)}{\alpha} dx_{T-k}^h
\]

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Thus, letting $\Delta := \frac{\alpha \beta(1+\alpha \beta)}{1-\alpha \beta} \left[ 1 - (\alpha \beta)^{k+1} \right]$, the FOC is given by

$$
\frac{(1 + \alpha \beta) \lambda^h (n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\alpha \beta \lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta}
$$

(I.25)

$$
+ \Delta \left[ \frac{\lambda^h (n_{T-k-1}) \beta}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} \right]
$$

$$
+ \frac{\Delta}{\alpha} \left( \mathbb{E}_{T-k-1} \left( (x^h_{T-k})^\alpha \right) \right)^{\frac{1}{\alpha}}
$$

$$
\int_{x^h_{T-k}}^{E_{T-k-1} (x^h_{T-k})^\alpha} \frac{\lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} \left( (x^h_{T-k})^\alpha \right) f \left( x^h_{T-k} \right)}{1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} \left( (x^h_{T-k})^\alpha \right)} dx^h_{T-k}
$$

$$
- \frac{\Delta}{\alpha} \left( \mathbb{E}_{T-k-1} \left( (x^h_{T-k})^\alpha \right) \right)^{\frac{1}{\alpha}}
$$

$$
\int_{x^h_{T-k}}^{E_{T-k-1} (x^h_{T-k})^\alpha} \frac{\lambda^h (n_{T-k-1}) f \left( x^h_{T-k} \right)}{1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} dx^h_{T-k} = 0
$$

Now, we have that

$$(1 + \alpha \beta) + \Delta = (1 + \alpha \beta) \left( \frac{1 - \alpha \beta + \alpha \beta \left[ 1 - (\alpha \beta)^{k+1} \right]}{1 - \alpha \beta} \right) = (1 + \alpha \beta) \left( \frac{1 - (\alpha \beta)^{k+2}}{1 - \alpha \beta} \right)
$$

(I.26)

Whilst

$$
\alpha \beta + \Delta = \alpha \beta \left( \frac{(1 - \alpha \beta) + (1 + \alpha \beta) \left[ 1 - (\alpha \beta)^{k+1} \right]}{1 - \alpha \beta} \right) = \alpha \beta \left( \frac{2 - (1 + \alpha \beta) (\alpha \beta)^{k+1}}{1 - \alpha \beta} \right)
$$

(I.27)

Thus, cancelling the common $\lambda^h (n_{T-k-1})$ term and rearranging gives the FOC for $\tau_{T-k-1}$ as

$$
\frac{(1 + \alpha \beta) \left( 1 - (\alpha \beta)^{k+2} \right)}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} - \frac{\alpha \beta \left( 2 - (1 + \alpha \beta) (\alpha \beta)^{k+1} \right)}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta}
$$

(I.28)
Thus, from the FOC, we must have combined terms being integrated are positive, and so the integral as a whole is positive. Hence the LHS is also positive. This expression can then be rearranged to show that the

\[ 1 - (\alpha \beta)^{k+1} \int_{x_{T-k}^h}^{(E_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\beta}}} \frac{(1 + \alpha \beta) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha) f(x_{T-k}^h)}{(x_{T-k}^h)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha)} dx_{T-k}^h = 0 \]

Note that the combined integral terms are positive. This follows as, over the range of the integral, \((E_{T-k-1}((x_{T-k}^h)^\alpha))^{\frac{1}{\beta}} > (x_{T-k}^h)\) and so \(E_{T-k-1}((x_{T-k}^h)^\alpha) > (x_{T-k}^h)^\alpha\). Thus

\[
E_{T-k-1}((x_{T-k}^h)^\alpha) \left( 1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \right) > (x_{T-k}^h)^\alpha - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1}((x_{T-k}^h)^\alpha) \tag{I.29}
\]

Given insolvency doesn’t occur under the conditions of the proposition, the RHS\(>0\), and hence the LHS is also positive. This expression can then be rearranged to show that the combined terms being integrated are positive, and so the integral as a whole is positive. Thus, from the FOC, we must have

\[
\frac{(1 + \alpha \beta) \left( 1 - (\alpha \beta)^{k+2} \right)}{1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} < \frac{\alpha \beta \left( 2 - (1 + \alpha \beta) (\alpha \beta)^{k+1} \right)}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta} \tag{I.30}
\]

To verify the FOC is indeed a maximum, we now turn to the second derivative of the planner’s objective w.r.t. \(\tau_{T-k-1}\). Ignoring constant terms that don’t affect the overall sign, it’s given by

\[
\frac{(1 + \alpha \beta) \left( 1 - (\alpha \beta)^{k+2} \right) \lambda^h (n_{T-k-1}) \beta}{(1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2} - \frac{\alpha \beta \left( 2 - (1 + \alpha \beta) (\alpha \beta)^{k+1} \right) \lambda^h (n_{T-k-1}) \beta}{\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2} \tag{I.31}
\]
We note that the integral is negative. Throughout the range of the integral we have, squaring both sides of (I.29) which maintains the inequality as both sides are positive,

\[
\left( \frac{1 - (\alpha \beta)^{k+1}}{(x_{T-k}^h)^{\alpha} - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} \right)^2 (1 + \alpha \beta) \lambda^h (n_{T-k-1}) f (x_{T-k}^h) \int_{x_{T-k}^h}^{x_{T-k}^h - k} \frac{dx_{T-k}^h}{(1 + (\alpha \beta)) (1 - (\alpha \beta)^{k+1})} \left( \frac{1 - (\alpha \beta)^{k+1}}{(x_{T-k}^h)^{\alpha} - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} \right)^2 \frac{dx_{T-k}^h}{(1 + \alpha \beta) \lambda^h (n_{T-k-1})^2}
\]

\[
\left( \frac{1 - (\alpha \beta)^{k+1}}{(x_{T-k}^h)^{\alpha} - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} \right)^2 (1 + \alpha \beta) \lambda^h (n_{T-k-1}) f (x_{T-k}^h) \int_{x_{T-k}^h}^{x_{T-k}^h - k} \frac{dx_{T-k}^h}{(1 + (\alpha \beta)) (1 - (\alpha \beta)^{k+1})} \left( \frac{1 - (\alpha \beta)^{k+1}}{(x_{T-k}^h)^{\alpha} - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})} \right)^2 \frac{dx_{T-k}^h}{(1 + \alpha \beta) \lambda^h (n_{T-k-1})^2}
\]

We note that the integral is negative. Throughout the range of the integral we have, squaring both sides of (I.29) which maintains the inequality as both sides are positive,

\[
(\mathbb{E}_{T-k-1} ((x_{T-k}^h)^{\alpha}))^2 (1 - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}))^2 > (x_{T-k}^h)^{\alpha} - (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \mathbb{E}_{T-k-1} ((x_{T-k}^h)^{\alpha})^2
\]

from which it follows, after rearranging, that the combined integral is negative. Now, noting that, as \(\pi < 1\),

\[
(1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^{-1} < (\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^{-1}
\]

and using (I.30), we have that, at the FOC

\[
\frac{(1 + \alpha \beta) (1 - (\alpha \beta)^{k+2}) \lambda^h (n_{T-k-1}) \beta}{(1 + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2} < \frac{\alpha \beta (2 - (1 + \alpha \beta) (\alpha \beta)^{k+1}) \lambda^h (n_{T-k-1}) \beta}{(\pi + (1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1}) \beta)^2}
\]

Hence, with the integral term in the second derivative negative, we have that the second derivative is negative at the FOC. Thus, the FOC as given by (I.28) is indeed an optimal solution.

It follows that the optimal effective pledgeability \((1 - \tau_{T-k-1}) \lambda^h (n_{T-k-1})\) is then equal to a constant, independent of \(k_{T-k-1}\). This completes the inductive step, hence by the Principle
of Mathematical Induction, \((1 - \tau_{T-s}) \lambda^h (n_{T-s})\) is independent \(k_{T-s} \forall s \geq 0\). Moreover, (I.9) follows from (I.28), when \(k = s - 1\), completing the proof.

I.3  Optimal policy with infinite horizon: Proposition 9

Proof of Proposition 9. Using Proposition I.1, and supposing \(T\) is infinitely far into the future, equivalently, \(s \to \infty\), we have, noting that \(\alpha \beta \in (0, 1)\) and so \(\lim_{s \to \infty} (\alpha \beta)^s = 0\), the FOC w.r.t \(\tau_{T-s}\) is given by

\[
\frac{(1 + \alpha \beta)}{1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} - \frac{2\alpha \beta}{\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} \tag{I.35}
\]

\[
\left(\mathbb{E}_{T-s}\left((x_{T-s+1}^h)^{\alpha}\right)\right)^{\frac{1}{\beta}} + \int_{x_{T-s+1}^h} \frac{1 + \alpha \beta \mathbb{E}_{T-s} \left((x_{T-s+1}^h)^{\alpha}\right) f \left(x_{T-s+1}^h\right)}{(x_{T-s+1}^h)^{\alpha} - (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \mathbb{E}_{T-s} \left((x_{T-s+1}^h)^{\alpha}\right)} dx_{T-s+1}^h = 0
\]

\[
- \int_{x_{T-s+1}^h} \frac{(1 + \alpha \beta) f \left(x_{T-s+1}^h\right)}{1 - (1 - \tau_{T-s}) \lambda^h (n_{T-s})} dx_{T-s+1}^h = 0
\]

Letting \(t := T - s\) and applying to sector \(h = a\) gives the result in the text. We next turn to the other parts of the proposition.

(i) It’s clear from the formula that effective pledgeability \((1 - \tau_t) \lambda^h (n_t)\), and hence effective leverage, is independent of \(k_t\) and constant across states. Thus, with \(\lambda^h (n_t)\) procyclical, we must have \(\tau_t\) also procyclical to ensure this is constant.

(ii) When there is no liquidation, with no shocks to \(x_{t+1}^h\), the integral term is zero (with zero support) and the FOC satisfies

\[
\frac{(1 + \alpha \beta)}{1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} = \frac{2\alpha \beta}{\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta} \tag{I.36}
\]
Rearranging this gives

\[(1 + \alpha \beta) (\pi + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta) = 2\alpha \beta \left(1 + (1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta\right)\]  (I.37)

\[(1 - \tau_{T-s}) \lambda^h (n_{T-s}) \beta (1 + \alpha \beta - 2\alpha \beta) = 2\alpha \beta - (1 + \alpha \beta) \pi\]  (I.38)

Effective pledgeability is then

\[(1 - \tau_{T-s}) \lambda^h (n_{T-s}) = \frac{2\alpha \beta - (1 + \alpha \beta) \pi}{\beta (1 - \alpha \beta)}\]  (I.39)

and rearranging gives the result in the text for effective leverage.

(iii) Let the FOC (56) be given by \(g(x_{t+1}^h, \tau_t (x_{t+1}^h)) \equiv 0\), where we consider the optimal policy as a function of the lower bound of the stochastic distribution, where we hold \(\mathbb{E}_t \left((x_{t+1}^h)^\alpha\right)\) fixed (e.g. by increasing the upper bound). Thus, when \(x_t^h = (\mathbb{E}_t \left((x_{t+1}^h)^\alpha\right))^{1/\alpha}\) we have the case of no shocks, and no liquidation. Consider a change to this lower bound.

Taking the total derivative of \(g(.,.)\) w.r.t. \(x_{t+1}^h\) we have

\[\frac{\partial g (x_{t+1}^h, \tau_t (x_{t+1}^h))}{\partial x_{t+1}^h} + \frac{\partial g (x_{t+1}^h, \tau_t (x_{t+1}^h))}{\partial \tau_t (x_{t+1}^h)} \frac{d\tau_t}{d x_{t+1}^h} = 0\]  (I.40)

and so

\[\frac{d\tau_t}{d x_{t+1}^h} = \frac{\partial g (x_{t+1}^h, \tau_t (x_{t+1}^h))}{\partial x_{t+1}^h} / -\frac{\partial g (x_{t+1}^h, \tau_t (x_{t+1}^h))}{\partial \tau_t (x_{t+1}^h)}\]  (I.41)

Now, from the proof of Proposition I.1 we have that \(\frac{\partial g (x_{t+1}^h, \tau_t (x_{t+1}^h))}{\partial \tau_t (x_{t+1}^h)} < 0\), with the second order condition for a maximum satisfied. Moreover, the proof of Proposition I.1 establishes that each term in the integral in the FOC, given by (56), is positive. Thus, it follows from Leibniz’s rule that \(\frac{\partial g (x_{t+1}^h, \tau_t (x_{t+1}^h))}{\partial x_{t+1}^h} < 0\), as positive terms are integrated over a smaller support. Combining these results we have that \(\frac{d\tau_t}{d x_{t+1}^h} < 0\). Hence, with \(\mathbb{E}_t \left((x_{t+1}^h)^\alpha\right)\) fixed, lowering the lower bound of the distribution, \(x_{t+1}^h\), and so increasing the range over which there is costly liquidation, raises the haircut \(\tau_t\), lowering effective leverage. Moreover, as
the case of no liquidation coincides with no shocks and \( x_t^h = \left( \mathbb{E}_t \left( (x_{t+1}^h)^\alpha \right) \right)^{\frac{1}{\alpha}} \), when there is liquidation, optimal effective leverage will be lower than when there is not.

This completes the proof of the proposition.

\[ \square \]

### J WELFARE SOLUTION WITH CREDIT TRAPS

#### J.1 Proposition 10

**Proof of Proposition 10.** Under the conditions of Proposition 10, and with the planner applying haircut \( \tau_t \), from (59) the trap threshold is defined implicitly by

\[
(\pi + \beta (1 - \tau_t) \lambda^a (\tilde{n}))^\alpha (1 + \beta (1 - \tau_t) \lambda^a (\tilde{n}))^{1-\alpha} \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) = (\pi + \beta (1 - \tau_t) \lambda^b)^\alpha (1 + \beta (1 - \tau_t) \lambda^b)^{1-\alpha} (x^b)^\alpha
\]

(J.1)

Note that, by assumption, \( \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) > (x^b)^\alpha \), and hence, we must have, with all expressions in brackets positive:

\[
(\pi + \beta (1 - \tau_t) \lambda^a (\tilde{n}))^\alpha (1 + \beta (1 - \tau_t) \lambda^a (\tilde{n}))^{1-\alpha} < (\pi + \beta (1 - \tau_t) \lambda^b)^\alpha (1 + \beta (1 - \tau_t) \lambda^b)^{1-\alpha}
\]

(J.2)

As the LHS is strictly increasing in \( \lambda^a (\tilde{n}) \), we must then have \( \lambda^a (\tilde{n}) < \lambda^b \).

We now turn to showing \( \frac{dn}{d\tau_t} < 0 \). Taking the logarithm of both sides of (59) gives

\[
alog (\pi + \beta (1 - \tau_t) \lambda^a (\tilde{n})) + (1 - \alpha) log (1 + \beta (1 - \tau_t) \lambda^a (\tilde{n})) = alog (\pi + \beta (1 - \tau_t) \lambda^b) + (1 - \alpha) log (1 + \beta (1 - \tau_t) \lambda^b) + D
\]

(J.3)
where \( D := \log \left( (x^b)^\alpha \right) - \log \left( \mathbb{E}_t \left( (x_{t+1}^a)^\alpha \right) \right) \) is independent of \( \tau_t \). Thus, with \( \tilde{n} \) a function of \( \tau_t \), \( \tilde{n} \) is implicitly defined by

\[
h (\tilde{n} (\tau_t) , \tau_t) \equiv D \tag{J.4}
\]

where

\[
h (\tilde{n} (\tau_t) , \tau_t) : = \alpha \left[ \log (\pi + \beta (1 - \tau_t) \lambda^a (\tilde{n})) - \log (\pi + \beta (1 - \tau_t) \lambda^b ) \right] + (1 - \alpha) \left( \log (1 + \beta (1 - \tau_t) \lambda^a (\tilde{n})) - \log (1 + \beta (1 - \tau_t) \lambda^b ) \right) \tag{J.5}
\]

Taking the total derivative of this w.r.t. \( \tau_t \) and rearranging, gives

\[
\frac{d \tilde{n}}{d \tau_t} = - \frac{\partial h (\tilde{n} (\tau_t) , \tau_t) }{\partial \tau_t} / \frac{\partial h (\tilde{n} (\tau_t) , \tau_t) }{\partial \tilde{n} (\tau_t)} \tag{J.6}
\]

Now

\[
\frac{\partial h (\tilde{n} (\tau_t) , \tau_t) }{\partial \tilde{n} (\tau_t)} = \left\{ \frac{\alpha \beta (1 - \tau_t) }{\left( \pi + \beta (1 - \tau_t) \lambda^a (\tilde{n}) \right)} + \frac{(1 - \alpha) \beta (1 - \tau_t) }{1 + \beta (1 - \tau_t) \lambda^a (\tilde{n})} \right\} \frac{d \lambda^a (\tilde{n})}{d \tilde{n}} > 0 \tag{J.7}
\]

as \( \frac{d \lambda^a (\tilde{n})}{d \tilde{n}} > 0 \) by Assumption 1' in the text. As this is non-zero, it also verifies the conditions for the Implicit Function Theorem.

Turning to the other term, we have

\[
- \frac{\partial h (\tilde{n} (\tau_t) , \tau_t) }{\partial \tau_t} = \alpha \left[ \frac{\beta \lambda^a (\tilde{n})}{\pi + \beta (1 - \tau_t) \lambda^a (\tilde{n})} - \frac{\beta \lambda^b}{\pi + \beta (1 - \tau_t) \lambda^b} \right] + (1 - \alpha) \left[ \frac{\beta \lambda^a (\tilde{n})}{1 + \beta (1 - \tau_t) \lambda^a (\tilde{n})} - \frac{\beta \lambda^b}{1 + \beta (1 - \tau_t) \lambda^b} \right] \tag{J.8}
\]

Rearranging gives

\[
- \frac{\partial h (\tilde{n} (\tau_t) , \tau_t) }{\partial \tau_t} = \alpha \beta \left[ \frac{\pi (\lambda^a (\tilde{n}) - \lambda^b)}{(\pi + \beta (1 - \tau_t) \lambda^a (\tilde{n})) (\pi + \beta (1 - \tau_t) \lambda^b)} \right] + (1 - \alpha) \beta \left[ \frac{(\lambda^a (\tilde{n}) - \lambda^b)}{(1 + \beta (1 - \tau_t) \lambda^a (\tilde{n})) (1 + \beta (1 - \tau_t) \lambda^b)} \right] \tag{J.9}
\]
As shown above, $\lambda^a(\tilde{n}) < \lambda^b$, and hence $-\frac{\partial h(\tilde{n}(\tau_t),\tau_t)}{\partial \tau_t} < 0$. It thus follows that $\frac{d\tilde{n}}{d\tau_t} < 0$, completing the proof.

K NUMERICAL WELFARE ANALYSIS - METHOD

We formulate the policy problem as a discrete time, continuous action, continuous state dynamic programming problem and apply the solution methods developed by Miranda and Fackler (2002), their dpsolve routine in particular. We formulate the policymaker’s problem as to solve the Bellman equation:

$$V(k_t) = \max_{\tau_t \in [0,1]} \left\{ W(k_t, \tau_t) + \beta V(k_{t+1}(k_t, \tau_t, x_{a,t+1})) \right\}$$

(K.1)

where $V(k_t)$ is the value function to be solved for, with an associated optimal policy rule as a function of the state of the economy, $\tau_t(k_t)$. An approximate solution to the Bellman equation can be found by constructing a value function approximant as a linear combination of a collection of basis functions whose coefficients are to be determined. The coefficients are determined, in turn, by requiring the approximant to satisfy the Bellman equation at $n$ collocation nodes. As described in Miranda and Fackler, this transforms the problem from one in which a functional equation must be solved for to one involving $n$ nonlinear equations in $n$ unknowns. This nonlinear problem can be solved using a number of different methods.

Practically this involves choices over how the stochastic process at work in the model economy is discretised and how the function space is approximated. Here there is generally a trade-off between approximation accuracy and computational efficiency. Having experimented with various formulations, the results we report in the text employ a 50-bin discretisation of the sector $a$ productivity shock process and a 20-point approximation to the value function. After convergence has been achieved, we compute the size of the residuals that remain between the left- and right-hand side of the Bellman equation at each value of
the state, evaluated using the value function approximants. The Bellman equation holds
exactly at the collocation nodes; we check that the residuals away from these nodes are
sufficiently small.

Figures K.1–K.4 show the optimised value function, policy rule, and Bellman equation
residuals for four specifications of the model: a baseline case with no liquidation and no
credit trap possible (Figure K.1); next a case adding in the possibility of liquidation (Figure
K.2); next a case in which there is no liquidation but credit traps are possible (Figure K.3);
and finally a case in which both liquidation and a credit trap are possible (Figure K.4). From
each of these Figures, note that the approximation residuals are at most of the order of $10^{-4}$,
an approximation tolerance we deem acceptable given the scale of the value function itself
(which is of the order $10^1$). We discuss the properties of the optimal policy rules shown in
panel (b) of each of the figure in more detail in the text.
Figure K.2: Value function, optimal policy, and approximation residuals: case with liquidation and no credit trap

Figure K.3: Value function, optimal policy, and approximation residuals: case of no liquidation with credit trap
L ALTERNATIVE PLEDGEABILITY CONSTRAINT FUNCTIONS

L.1 Pledgeability constraint function of bank equity to assets

Proposition L.1. Suppose the conditions of Proposition 3 hold. Suppose that instead of being a function of the current health of the banking system, \( n_t \), the pledgeability constraint \( \lambda^h(.) \) is a function of the ratio of banking net worth to the expected value of assets held by banks. That is, the pledgeability constraint is given by \( \lambda^h \left( \frac{n_t}{\Gamma_t} \right) \), where \( \Gamma_t \) is the expected value of assets held by banks. The pledgeability constraint \( \lambda^h(.) \) is strictly increasing, so the pledgeability constraint is loosened when the ratio of banking net worth to assets is higher. Suppose further that \( \lambda^h(.) \) is taken as given by banks.

Then the solution to this model is as with the baseline model except with the function \( \lambda^h(n_t) \) replaced throughout by a function \( \tilde{\lambda}^h(n_t) \), where \( \tilde{\lambda}^h(n_t) \) satisfies \( \tilde{\lambda}^h(n_t) \in (0, 1) \) and
Thus having $\lambda^h()$ be a function of $n_t$ in the baseline model, rather than the ratio of banking net worth to assets, is without loss of generality.

**Proof.** To simplify the notation we drop the $h$ superscript throughout the proof. Let $\Gamma_t$ be the expected value of assets held by banks. From equation (E.2), taking time $t$ expectations, we have, given $\lambda(.)$:

$$\Gamma_t := \alpha E_t ((x_{t+1})^{\alpha}) \left( \frac{\pi + \beta \lambda(.)}{1 + \beta \lambda(.)} \right)^{\alpha} \left( 1 - \alpha \frac{n_t}{\pi} \right)^{\alpha}$$  \hspace{1cm} (L.1)

Thus, the ratio $\frac{n_t}{\Gamma_t}$ is given by

$$\frac{n_t}{\Gamma_t} = \frac{n_t^{-\alpha}}{\alpha E_t ((x_{t+1})^{\alpha}) \left( \frac{(1-\alpha)}{\pi} \right)^{\alpha} \left( \frac{\pi + \beta \lambda(.)}{1 + \beta \lambda(.)} \right)^{\alpha}}$$  \hspace{1cm} (L.2)

Thus, with $\lambda(.)$ an increasing function of $\frac{n_t}{\Gamma_t}$, the following equation implicitly defines $\frac{n_t}{\Gamma_t}$:

$$\frac{n_t}{\Gamma_t} = \frac{n_t^{-\alpha}}{\alpha E_t ((x_{t+1})^{\alpha}) \left( \frac{(1-\alpha)}{\pi} \right)^{\alpha} \left( \frac{\pi + \beta \lambda\left(\frac{n_t}{\Gamma_t}\right)}{1 + \beta \lambda\left(\frac{n_t}{\Gamma_t}\right)} \right)^{\alpha}}$$  \hspace{1cm} (L.3)

We proceed in two steps.

**Step 1:** For each $n_t > 0$, there is a unique function $z(n_t) = \frac{n_t}{\Gamma_t} > 0$ that solves equation (L.3).

We first show existence. Let

$$g(z) := z - \frac{n_t^{-\alpha}}{\alpha E_t ((x_{t+1})^{\alpha}) \left( \frac{(1-\alpha)}{\pi} \right)^{\alpha} \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^{\alpha}}$$  \hspace{1cm} (L.4)

We show that $\exists z > 0 : g(z) = 0$. For given $n_t > 0$, we have

$$\lim_{z \to 0} g(z) = 0 - \frac{n_t^{-\alpha}}{\alpha E_t ((x_{t+1})^{\alpha}) \left( \frac{(1-\alpha)}{\pi} \right)^{\alpha} \left( \frac{\pi + \beta \lambda(0)}{1 + \beta \lambda(0)} \right)^{\alpha}} < 0$$  \hspace{1cm} (L.5)
as $\lambda (.) \in (0, 1)$ and $n_t > 0$.

Now, as $\lambda (.) \in (0, 1)$,

$$
\frac{n_t^{1-\alpha}}{\alpha \mathbb{E}_t ((x_{t+1})^\alpha) \left( \frac{1-\alpha}{\pi} \right)^\alpha \left( \frac{\pi+\beta \lambda(z)}{1+\beta \lambda(z)} \right)^\alpha} \quad (L.6)
$$

is bounded as $z \to \infty$. Hence, $g(z) \to 0$ as $z \to \infty$. Thus, for sufficiently large $z$, $g(z) > 0$.

Hence, as $g(.)$ is continuous, by the Intermediate Value Theorem, $\exists z : g(z) = 0$. Moreover, as $\lim_{z \to 0} g(z) < 0$, $\exists z > 0 : g(z) = 0$. Thus, there exists at least one $\frac{n_t}{\Gamma_t} > 0$ that solves (L.3).

We now turn to uniqueness. We show that $g'(z) > 0$, ensuring there is a unique $z > 0 : g(z) = 0$.

$$
g'(z) = 1 - \frac{n_t^{1-\alpha}}{\alpha \mathbb{E}_t ((x_{t+1})^\alpha) \left( \frac{1-\alpha}{\pi} \right)^\alpha \left( \frac{\pi+\beta \lambda(z)}{1+\beta \lambda(z)} \right)^\alpha} d \left\{ \left( \frac{1 + \beta \lambda(z)}{\pi + \beta \lambda(z)} \right)^\alpha \right\} \quad (L.7)
$$

Well,

$$
\frac{d}{dz} \left\{ \left( \frac{1 + \beta \lambda(z)}{\pi + \beta \lambda(z)} \right)^\alpha \right\} = \alpha \left( \frac{1 + \beta \lambda(z)}{\pi + \beta \lambda(z)} \right)^{\alpha-1} \beta \lambda'(z) \left( \frac{\pi - 1}{(\pi + \beta \lambda(z))^2} \right) < 0 \quad (L.8)
$$

where the inequality follows as $\pi < 1$ and $\lambda'(z) > 0$. Thus, we have that $g'(z) > 0$, ensuring there is a unique $z > 0 : g(z) = 0$. We can thus write this solution as a function $z(n_t)$. This completes the proof of Step 1.

**Step 2:** Let $z(n_t) = \frac{n_t}{\Gamma_t} > 0$ be the unique positive solution to (L.3). Then $z'(n_t) > 0$.

To show this, we utilise $g(z(n_t), n_t)$, as defined above, making the dependence on $n_t$ explicit, where $z(n_t)$ is the unique positive solution for given $n_t$. Then, we have

$$
g(z(n_t), n_t) \equiv 0 \quad (L.9)
$$

Total differentiating w.r.t. $n_t$, and rearranging gives, via the Implicit Function Theorem:

$$
z'(n_t) = -\frac{\partial g(z(n_t), n_t)}{\partial n_t} \frac{\partial g(z(n_t), n_t)}{\partial z} \quad (L.10)
$$
Now, as shown in Step 1, \( \frac{\partial g(z(n_t), n_t)}{\partial z} > 0 \). Moreover,

\[
\frac{\partial g(z(n_t), n_t)}{\partial n_t} = -\frac{(1 - \alpha) n_t^{-\alpha}}{\alpha E_t \left( (x_{t+1})^\alpha \right) \left( \frac{(1 - \alpha)}{\pi} \right)^{\alpha} \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^{\alpha}} < 0 \tag{L.11}
\]

Hence, \( z'(n_t) > 0 \). This completes the proof of Step 2.

Thus, we can define a function

\[
\tilde{\lambda}(n_t) := \lambda(z(n_t)) = \lambda \left( \frac{n_t}{\Gamma_t} \right) \tag{L.12}
\]

and we have \( \tilde{\lambda}'(n_t) = \lambda'(z(n_t)) z'(n_t) > 0 \). Note that as \( z(n_t) > 0 \) and \( \lambda(.) \in (0, 1) \), we have \( \tilde{\lambda}(n_t) \in (0, 1) \). This completes the proof of the proposition.

\[
\square
\]

L.2 Pledgeability constraint function of expected future banking net worth

**Proposition L.2.** Suppose the conditions of Proposition 3 hold. Moreover suppose that

\[
\lambda^h(z) > \frac{\alpha \beta (1 - \pi) - \pi}{\alpha \beta (1 - \pi) + \beta (1 + \pi)} \tag{L.13}
\]

for all \( z > 0 \). Suppose that instead of being a function of the current health of the banking system, \( n_t \), the pledgeability constraint \( \lambda^h(.) \) is a function of \( \mathbb{E}_t V^h_{t+1}(n_t) \), the expected value of the current generation of banks at the start of the next period, following the realisation of the technology shock. The pledgeability constraint \( \lambda^h(.) \) is strictly increasing, so the constraint is loosened when the expected value of the banking system next period is higher. Suppose further that \( \lambda^h(.) \) is taken as given by banks.

Then the solution to this model is as with the baseline model except with the function \( \lambda^h(n_t) \) replaced throughout by a function \( \tilde{\lambda}^h(n_t) \), where \( \tilde{\lambda}^h(n_t) \) satisfies \( \tilde{\lambda}^h(n_t) \in (0, 1) \) and \( \frac{d\tilde{\lambda}^h(n_t)}{dn_t} > 0 \). Thus having \( \lambda^h(.) \) be a function of \( n_t \) in the baseline model, rather than
\( \mathbb{E}_t V^h_{t+1}(n_t) \), is without loss of generality.

**Proof.** To simplify the notation we drop the \( h \) superscript throughout the proof. At time \( t \), \( \lambda \) is a function of the expected value of the current generation of banks in the following period, following the realisation of the technology shock: \( \lambda(\mathbb{E}_t V_{t+1}(n_t)) \), where \( \mathbb{E}_t V_{t+1}(n_t) \) is a function of the aggregate banking system net worth in the current period, \( n_t \). From equation (E.7), with individual banks taking \( \lambda \) as given, we have the following identity that implicitly defines \( \mathbb{E}_t V_{t+1}(n_t) \):

\[
\mathbb{E}_t V_{t+1}(n_t) = \alpha \mathbb{E}_t ((x_{t+1})^\alpha) \left[ 1 - \lambda(\mathbb{E}_t V_{t+1}(n_t)) \right] \left( \frac{\pi + \beta \lambda(n_t)}{1 + \beta \lambda(n_t)} \right)^\alpha \frac{n_t}{\pi} \quad \text{(L.14)}
\]

We proceed in two steps.

**Step 1:** For each \( n_t > 0 \), there is a unique function \( z(n_t) = \mathbb{E}_t V_{t+1}(n_t) > 0 \) that solves equation (L.14).

We first show existence. Let

\[
g(z) := z - \alpha \mathbb{E}_t ((x_{t+1})^\alpha) \left[ 1 - \lambda(z) \right] \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^\alpha \frac{n_t}{\pi} \quad \text{(L.15)}
\]

We show that \( \exists z > 0 : g(z) = 0 \). For given \( n_t > 0 \), we have

\[
g(0) = 0 - \alpha \mathbb{E}_t ((x_{t+1})^\alpha) \left[ 1 - \lambda(0) \right] \left( \frac{\pi + \beta \lambda(0)}{1 + \beta \lambda(0)} \right)^\alpha \frac{n_t}{\pi} < 0 \quad \text{(L.16)}
\]

as \( \lambda(.) \in (0, 1) \) and \( n_t > 0 \).

Now, as \( \lambda(.) \in (0, 1) \),

\[
[1 - \lambda(z)] \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^\alpha \frac{n_t}{\pi} \quad \text{(L.17)}
\]

is bounded as \( z \to \infty \). Hence, \( g(z) \to \infty \) as \( z \to \infty \). Thus, for sufficiently large \( z \), \( g(z) > 0 \). Hence, as \( g(.) \) is continuous, by the Intermediate Value Theorem, \( \exists z : g(z) = 0 \). Moreover,
as \( g(0) < 0, \exists z > 0 : g(z) = 0 \). Thus, there exists at least one \( \mathbb{E}_t V_{t+1}^h (n_t) > 0 \) that solves (L.14).

We now turn to uniqueness. We show that \( g'(z) > 0 \), ensuring there is a unique \( z > 0 : g(z) = 0 \).

\[
g'(z) = 1 - \alpha \mathbb{E}_t ((x_{t+1})^\alpha) \frac{d}{dz} \left\{ \left[ 1 - \lambda(z) \right] \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^\alpha \right\} \quad \text{(L.18)}
\]

We show that \( \frac{d}{dz} \{ . \} < 0 \). \( \frac{d}{dz} \{ . \} \) is given by

\[
-\lambda'(z) \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^\alpha + [1 - \lambda(z)] \alpha \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^{\alpha-1} \left( \frac{\beta \lambda'(z) [(1 + \beta \lambda(z)) - (\pi + \beta \lambda(z))]}{(1 + \beta \lambda(z))^2} \right)
\]

This can be simplified to

\[
-\lambda'(z) \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^\alpha \left\{ 1 - \frac{(1 - \lambda(z)) \alpha \beta (1 - \pi)}{\left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right) (1 + \beta \lambda(z))} \right\} \quad \text{(L.20)}
\]

As \(-\lambda'(z) \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right)^\alpha < 0\), it’s then sufficient to prove that

\[
\frac{(1 - \lambda(z)) \alpha \beta (1 - \pi)}{\left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right) (1 + \beta \lambda(z))} < 1
\]

(L.21)

And with the denominator positive, this holds iff

\[
(1 - \lambda(z)) \alpha \beta (1 - \pi) < (\pi + \beta \lambda(z)) (1 + \beta \lambda(z))
\]

(L.22)

Expanding, this condition can be written as

\[
\lambda^2(z) (\beta^2) + \lambda(z) (\alpha \beta (1 - \pi) + \beta (1 + \pi)) > \alpha \beta (1 - \pi) - \pi
\]

(L.23)

This holds given that the coefficient on \( \lambda^2(z) \) is positive, and by the condition of the
proposition,

\[ \lambda(z) > \frac{\alpha \beta (1 - \pi) - \pi}{\alpha \beta (1 - \pi) + \beta (1 + \pi)} \quad (L.24) \]

Thus, we have that \( g'(z) > 0 \), ensuring there is a unique \( z > 0 : g(z) = 0 \). We can thus write this solution as a function \( z(n_t) \). This completes the proof of Step 1.

**Step 2**: Let \( z(n_t) = E_t V_{t+1}(n_t) > 0 \) be the unique positive solution to (L.14). Then \( z'(n_t) > 0 \).

To show this, we utilise \( g(z(n_t), n_t) \), as defined above, making the dependence on \( n_t \) explicit, where \( z(n_t) \) is the unique positive solution for given \( n_t \). Following the steps of the immediately prior proof,

\[
z'(n_t) = \frac{\partial g(z(n_t), n_t)}{\partial n_t} \frac{\partial g(z(n_t), n_t)}{\partial z} \quad (L.25)
\]

Now, as shown in Step 1, \( \frac{\partial g(z(n_t), n_t)}{\partial z} > 0 \). Moreover,

\[
\frac{\partial g(z(n_t), n_t)}{\partial n_t} = -\alpha E_t ((x_{t+1})^\alpha) [1 - \lambda(z)] \left( \left( \frac{\pi + \beta \lambda(z)}{1 + \beta \lambda(z)} \right) \frac{1}{\pi} \right)^\alpha \alpha n_t^{\alpha-1} < 0 \quad (L.26)
\]

Hence, \( z'(n_t) > 0 \). This completes the proof of Step 2.

Thus, we can define a function

\[
\tilde{\lambda}(n_t) := \lambda(z_t(n_t)) = \lambda(E_t V_{t+1}(n_t)) \quad (L.27)
\]

and we have \( \tilde{\lambda}'(n_t) = \lambda'(z(n_t))z'(n_t) > 0 \). Note that as \( z(n_t) > 0 \) and \( \lambda(.) \in (0,1) \), we have \( \tilde{\lambda}(n_t) \in (0,1) \). This completes the proof of the proposition.

\( \Box \)
References